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Some Bounds for The Energies of The Power Graphs of Cyclic Groups

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Cyclic group; Power graph; Energy; Bound

Abstract

In this study, some lower and upper bounds were obtained for the energies of the power graphs of finite cyclic groups by considering the adjacency matrix structure of the power graph of a finite cyclic group. Then, some results are given using the relationship between the case where the order of a cyclic group is the positive integer power of a prime number and the completeness of the power graph corresponding to this cyclic group.

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Devirli Grupların Power Graflarının Enerjileri İçin Bazı Sınırlar

Anahtar kelimeler
Devirli grup; Power graf; Enerji; Sınır

Öz

Bu çalışmada, sonlu bir devirli grubun power grafının komşuluk matrisi yapısı dikkate alınarak, sonlu devirli grupların power graflarının enerjileri için bazı alt ve üst sınırlar elde edilmiştir. Daha sonra devirli bir grubun mertebesinin bir asal sayının pozitif tam sayı kuvveti olması durumu ile bu devirli gruba karşılık gelen power grafın tamlığı arasındaki ilişki kullanılarak bazı sonuçlar verilmiştir.

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1. Introduction and Preliminaries

Let G be a group. The undirected power graph $P(G)$ of G is an undirected graph whose vertices are represented by the elements of G and whose adjacency relation between any two vertices i and j are given by

$$i \sim j \Leftrightarrow i \neq j, i = j^m \text{ or } j = i^m,$$

where $m \in \mathbb{Z}^+$. The concept of power graph was first defined by Kelarev and Quinn in 2000 as a directed power graph of finite semigroups (Kelarev and Quinn 2000). Later, the concept of directed power graph for groups was defined, and various studies were carried out for directed power graphs on semigroups and groups (Kelarev and Quinn 2000, Kelarev and Quinn 2002, Kelarev et al. 2004). Chakrabarty et al., inspired by these studies, introduced the concept of undirected power graph to the mathematical literature with their study in 2009 (Chakrabarty et al. 2009). In addition, in this study, they showed that for the power graph to be a complete graph, the order of the group must be 1 or a positive integer power of a prime number. Later, Cameron et al. shortened the concept of undirected power graph and named it as power graph and this name passed into the mathematics literature and after that, studies for undirected power graphs were published under the name of power graph (Cameron 2010, Cameron and Ghosh

2011, Chattopadhyay and Panigrahi 2014, Chattopadhyay and Panigrahi 2015).

Another name that directs the study of power graph in terms of spectral graph theory is Chattopadhyay. In the study of Chattopadhyay et al. in 2018, the adjacency matrix concept was redefined for power graphs and the bounds for the largest eigenvalues of the adjacency matrices of the power graphs were obtained (Chattopadhyay et al. 2018).

The concept of energy, which has an important place in spectral graph theory, was introduced to the mathematical literature by Gutman in 1978 (Gutman 1978). According to Gutman's definition, the energy of a graph is the sum of the absolute values of the eigenvalues of the graph's adjacency matrix. In the past decades, the energy of a graph has received much interest and many different versions have been conceived. The most popular and most studied energy versions are the Laplacian energy (Gutman and Zhou 2006), the signless Laplacian energy (Abreu et al. 2010), the distance energy (Gutman et al 2008) and the Randic energy (Cavers et al. 2010). It is important to do boundary studies for energy as it is not always easy to calculate the energy of a graph. Therefore, boundary studies for energy have attracted a lot of attention and many studies have been carried out in this field.

In this study, inspired by the concept of adjacency matrix of the power graph on a cyclic group, bounds for the energy of the power graph on a finite cyclic group C_n of order n were obtained. Throughout this study, the set of the identity and generators of C_n is denoted as V_1 and the set of other elements as V_2. Also $|V_1| = 1 + \varphi(n) = \ell$ (say), where $\varphi(n)$ is Euler's φ function. Then the adjacency matrix A of the power graph $P(C_n)$ is of the form

$$A = \begin{pmatrix} J_{\ell\times\ell} - I_{\ell\times\ell} & J_{\ell\times(n-\ell)} \\ J_{(n-\ell)\times\ell} & A\big(\mathcal{P}(V_2)\big)_{(n-\ell)\times(n-\ell)} \end{pmatrix},$$

where, I and J are the unit matrix and the all-ones matrix, respectively. Also $A\big(P(V_2)\big)$ is the adjacency matrix of the power graph induced by the vertex set V_2. The adjacency matrix is a real symmetric matrix and its eigenvalues denoted by

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

In addition, considering the energy concept, the energies of the power graphs of finite cyclic groups are defined as the sum of the absolute values of the eigenvalues of the adjacency matrices, i.e.,

$$E = \sum_{i=1}^{n} |\lambda_i|.$$

2. Sharp Bounds for The Energy of $P(C_n)$

In this section, some upper and lower bounds for the energy of the power graph of C_n are obtained. In addition, results are given for cases where the power graphs of finite cyclic groups are complete graphs, considering the order of the cyclic group. The following lemma will help to obtain some bounds for the energy of $P(C_n)$ using the trace of the square of the adjacency matrix of $P(C_n)$.

Lemma 2.1. Let $P(C_n)$ be the power graph of C_n with $n \geq 3$. Then

$$\sum_{i=1}^{n} \lambda_i = 0$$

and

$$\sum_{i=1}^{n} \lambda_i^2 = \ell(2n - \ell - 1) + \ 2\sum_{\ell+1\leq i<j\leq n} a_{ij}^2.$$

Proof. From the definition of the trace, we have

$$\sum_{i=1}^{n} \lambda_i = tr(A) = 0.$$

We now consider the matrix A^2.

$$\sum_{i=1}^{n} \lambda_i^2 = tr\,(A^2)$$

$$= \ell(\ell-1) + \ell(n-\ell) + \ell(n-\ell) + tr\big[A^2\big(\mathcal{P}(V_2)\big)\big]$$

$$= \ell(2n - \ell - 1) + + tr\big[A^2\big(\mathcal{P}(V_2)\big)\big].$$

The *ii*-th entry of $\big[A\big(P(V^2)\big)\big]^2$ is $\sum_{\substack{j=\ell+1 \\ j\neq i}}^{n} a_{ij}^2$. Thus

$$\sum_{i=1}^{n} \lambda_i^2 = \ell(2n - \ell - 1) + tr\big[A^2\big(\mathcal{P}(V_2)\big)\big]$$

$$= \ell(2n - \ell - 1) + 2\sum_{\ell+1\leq i<j\leq n} a_{ij}^2.$$

This completes the proof.

Lemma 2.2. Let $P(C_n)$ be the power graph of a cyclic group C_n, whose order is a positive integer power of a prime number and $n \geq 3$. Then

$\sum_{i=1}^{n} \lambda_i^2 = n(n-1)$.

Proof. We know that if C_n is a cyclic group, whose order is a positive integer power of a prime number then $P(C_n)$ is a complete graph and $|V_1| = \ell = n$ and $|V_2| = 0$. Using Lemma 2.1, we have

$\sum_{i=1}^{n} \lambda_i^2 = tr(A^2)$

$= \ell(2n-\ell-1) + 2\sum_{\ell+1 \leq i<j \leq n} a_{ij}^2$

$= n(2n-n-1) = n(n-1)$.

Thus, the proof is completed.

Lemma 2.3. Let $a, b, x, y \in \mathbb{R}^+$ and $0 < a \leq x \leq y \leq b$. Then

$\frac{\sqrt{xy}}{x+y} \geq \frac{\sqrt{ab}}{a+b}$

(Oboudi 2019).

Theorem 2.1. Let E be the energy of the power graph $P(C_n)$. Then

$$\sqrt{\ell(2n-\ell-1) + 2\sum_{\ell+1 \leq i<j \leq n} a_{ij}^2} \leq E$$

and

$$E \leq \sqrt{n[\ell(2n-\ell-1) + 2\sum_{\ell+1 \leq i<j \leq n} a_{ij}^2]}\,.$$

Proof. Using the Cauchy-Schwarz inequality and Lemma 2.1, we have

$E^2 = (\sum_{i=1}^{n} |\lambda_i|)^2 \leq n\sum_{i=1}^{n} \lambda_i^2$

$= n\big[\ell(2n-\ell-1) + 2\sum_{\ell+1 \leq i<j \leq n} a_{ij}^2\big]$. (1)

For the proof of the first inequality, we have

$E^2 = (\sum_{i=1}^{n} |\lambda_i|)^2 \geq \sum_{i=1}^{n} \lambda_i^2$

$= \ell(2n-\ell-1) + 2\sum_{\ell+1 \leq i<j \leq n} a_{ij}^2$. (2)

By using (1) and (2), the required result is obtained.

Corollary 2.1. Let $P(C_n)$ be the power graph of a cyclic group C_n, whose order is a positive integer power of a prime number and $n \geq 3$. Then

$\sqrt{n(n-1)} \leq E \leq n\sqrt{n-1}\,.$

Proof. Since C_n is a cyclic group, whose order is a positive integer power of a prime number, $P(C_n)$ is a complete graph. Using Theorem 2.1 and Lemma 2.2, we have

$E^2 = (\sum_{i=1}^{n} |\lambda_i|)^2$

$\leq n\sum_{i=1}^{n} \lambda_i^2 = n^2(n-1)$,

and then

$E \leq n\sqrt{n-1}\,.$

For the proof of the first inequality, we have

$E^2 = (\sum_{i=1}^{n} |\lambda_i|)^2$

$\geq \sum_{i=1}^{n} \lambda_i^2 = n(n-1)$,

i.e.,

$E \geq \sqrt{n(n-1)}$.

Thus, the proof is complete.

Theorem 2.2. Let $P(C_n)$ be the power graph of C_n with $n \geq 3$ and λ_1 is largest eigenvalue of $P(C_n)$. Then

$E \leq \lambda_1 + \sqrt{(n-1)(s-\lambda_1^2)}$,

where $s = \ell(2n-\ell-1) + 2\sum_{\ell+1 \leq i<j \leq n} a_{ij}^2$.

Proof. Using Lemma 2.1 and the Cauchy-Schwarz inequality, we have

$(E-\lambda_1)^2 = (\sum_{i=2}^{n} |\lambda_i|)^2$

$\leq (n-1)\big(\sum_{i=1}^{n} \lambda_i^2 - \lambda_1^2\big)$

$= (n-1)\big[\ell(2n-\ell-1) + 2\sum_{\ell+1 \leq i<j \leq n} a_{ij}^2 - \lambda_1^2\big]$.

Since

$s = \ell(2n - \ell - 1) + 2\sum_{\ell+1 \le i<j \le n} a_{ij}^2,$

$E \le \lambda_1 + \sqrt{(n-1)(s - \lambda_1^2)}.$

Thus, the required result is obtained.

Corollary 2.2. Let $P(C_n)$ be the power graph of a cyclic group C_n, whose order is a positive integer power of a prime number and $n \ge 3$. Then

$E = 2(n-1).$

Proof. Since C_n is a cyclic group, whose order is a positive integer power of a prime number, $P(C_n)$ is a complete graph and $\lambda_1 = n - 1$. By Theorem 2.2 and Lemma 2.2, we obtain

$E \le \lambda_1 + \sqrt{(n-1)(s - \lambda_1^2)}$

$= n - 1 + \sqrt{(n-1)(n(n-1) - (n-1)^2)}$

$= 2(n-1).$

The proof is complete.

Theorem 2.3. Let $P(C_n)$ be the power graph of C_n with $n \ge 3$. Then

$E \ge \sqrt{s + n(n-1)detA^{\frac{2}{n}}},$

where $s = \ell(2n - \ell - 1) + 2\sum_{\ell+1 \le i<j \le n} a_{ij}^2$.

Proof. By Lemma 2.1, we have

$E^2 = \left(\sum_{i=1}^n |\lambda_i|\right)^2$

$= \sum_{i=1}^n \lambda_i^2 + 2\sum_{1 \le i<j \le n} |\lambda_i||\lambda_j|$

$= \ell(2n - \ell - 1) + 2\sum_{\ell+1 \le i<j \le n} a_{ij}^2 +$
$\sum_{i \ne j} |\lambda_i||\lambda_j|$

$= s + \sum_{i \ne j} |\lambda_i||\lambda_j|$ (3)

Since the geometric mean of nonnegative numbers is smaller than their arithmetic mean. Thus, we have

$\frac{1}{n(n-1)}\sum_{i \ne j} |\lambda_i||\lambda_j| \ge \left(\prod_{i \ne j} |\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}$

$= \left(\prod_{i=1}^n |\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$

$= \prod_{i=1}^n |\lambda_i|^{\frac{2}{n}}$

$= detA^{\frac{2}{n}}$ (4)

By (3) and (4), we obtain

$E \ge \sqrt{s + n(n-1)detA^{\frac{2}{n}}}.$

Corollary 2.3. Let $P(C_n)$ be the power graph of a cyclic group C_n, whose order is a positive integer power of a prime number and $n \ge 3$. Then

$E \ge \sqrt{n(n-1)\left[1 + (n-1)^{\frac{2}{n}}\right]}.$

Proof. Since $|C_n| = p^m$, $\mathcal{P}(C_n)$ is a complete graph and its spectrum is $\left\{n-1, \underbrace{-1, -1, \dots, -1}_{n-1}\right\}$. Using Theorem 2.3 and Lemma 2.2, we obtain

$E \ge \sum_{i=1}^n \lambda_i^2 + n(n-1)detA^{\frac{2}{n}}$

$= n(n-1) + n(n-1)[(n-1)(-1)^{n-1}]^{\frac{2}{n}}$

$= n(n-1)\left[1 + (n-1)^{\frac{2}{n}}\right]$

and then

$E \ge \sqrt{n(n-1)\left[1 + (n-1)^{\frac{2}{n}}\right]}.$

The proof is completed.

Theorem 2.4. Let $P(C_n)$ be the power graph of C_n with $n \ge 3$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of the adjacency matrix of $P(C_n)$, with $|\lambda_1| \ge |\lambda_2| \ge \dots \ge |\lambda_n| \ge 0$. Then

$E \ge \frac{s + n|\lambda_1||\lambda_n|}{|\lambda_1| + |\lambda_n|},$

where $s = \ell(2n - \ell - 1) + 2\sum_{\ell+1 \le i<j \le n} a_{ij}^2$.

Proof. Since C_n is a cyclic group with $n \ge 3$, the power graph $\mathcal{P}(C_n)$ has at least two edges. Therefore, the adjacency matrix of $\mathcal{P}(C_n)$ has at least one nonzero eigenvalue. Now for $i = 1,2,\ldots,n$ be the $|\lambda_1| \geq |\lambda_i| \geq |\lambda_n|$. In this situation,

$$(|\lambda_1| - |\lambda_i|)(|\lambda_i| - |\lambda_n|) \geq 0. \qquad (5)$$

On the other hand,

$$(|\lambda_1| - |\lambda_i|)(|\lambda_i| - |\lambda_n|) = |\lambda_i|(|\lambda_1| + |\lambda_n|) - (\lambda_i^2 + |\lambda_1||\lambda_n|). \qquad (6)$$

By (5) and (6), we obtain

$$|\lambda_i|(|\lambda_1| + |\lambda_n|) \geq \lambda_i^2 + |\lambda_1||\lambda_n|.$$

If this operation is applied for every i, we get

$$(|\lambda_1| + |\lambda_2| + \ldots + |\lambda_n|)(|\lambda_1| + |\lambda_n|) \geq (\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2) + n|\lambda_1||\lambda_n|.$$

Also using Lemma 2.1, we have

$$E \geq \frac{s + n|\lambda_1||\lambda_n|}{|\lambda_1| + |\lambda_n|}.$$

Therefore, the proof is completed.

Corollary 2.4. Let $P(C_n)$ be the power graph of a cyclic group C_n, whose order is a positive integer power of a prime number and $n \geq 3$. Then

$$E = 2(n-1).$$

Proof. Since $|C_n| = p^m$, $\mathcal{P}(C_n)$ is a complete graph and its spectrum is $\left\{n-1, \underbrace{-1,-1,\ldots,-1}_{n-1}\right\}$.

For $i = 2,3,\ldots,n$

$$|\lambda_1| \geq |\lambda_i| = |\lambda_2| = \ldots = |\lambda_n|,$$

and thus

$$(|\lambda_1| - |\lambda_i|)(|\lambda_i| - |\lambda_n|) = 0.$$

Using Theorem 2.3 and Lemma 2.2, we obtain

$$E = \frac{\sum_{i=1}^{n} \lambda_i^2 + n|\lambda_1||\lambda_n|}{|\lambda_1| + |\lambda_n|}$$

$$= \frac{n(n-1) + n(n-1)}{1 + n - 1}$$

$$= 2(n-1).$$

Thus, the required result is obtained.

Theorem 2.5. Let $P(C_n)$ be the power graph of C_n with $n \geq 3$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the adjacency matrix of $P(C_n)$, with $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| > 0$. Then

$$E \geq \frac{2\sqrt{|\lambda_1||\lambda_n|}\sqrt{ns}}{|\lambda_1| + |\lambda_n|},$$

where $s = \ell(2n - \ell - 1) + 2\sum_{\ell+1 \leq i < j \leq n} a_{ij}^2$.

Proof Since C_n is a cyclic group with $n \geq 3$, the power graph $\mathcal{P}(C_n)$ has at least two edges. Therefore, the adjacency matrix of $\mathcal{P}(C_n)$ has at least one nonzero eigenvalue. Using Theorem 2.4, we get

$$E \geq \frac{\sum_{i=1}^{n} \lambda_i^2 + n|\lambda_1||\lambda_n|}{|\lambda_1| + |\lambda_n|} \qquad (7)$$

On the other hand,

$$a + b \geq 2\sqrt{ab}$$

for $a, b \geq 0$. Thus

$$\sum_{i=1}^{n} \lambda_i^2 + n|\lambda_1||\lambda_n| \geq 2\sqrt{|\lambda_1||\lambda_n|}\sqrt{n\sum_{i=1}^{n} \lambda_i^2}. \qquad (8)$$

Using (7), (8) and Lemma 2.1, we have

$$E \geq \frac{2\sqrt{|\lambda_1||\lambda_n|}\sqrt{n\sum_{i=1}^{n} \lambda_i^2}}{|\lambda_1| + |\lambda_n|} = \frac{2\sqrt{|\lambda_1||\lambda_n|}\sqrt{ns}}{|\lambda_1| + |\lambda_n|}.$$

The proof is completed.

Theorem 2.6. Let $P(C_n)$ be the power graph of C_n with $n \geq 3$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the adjacency matrix of $P(C_n)$, with

$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \geq 1$. Then

$$E \geq 2\sqrt{\frac{n-1}{n}s},$$

where $s = \ell(2n - \ell - 1) + 2\sum_{\ell+1 \leq i < j \leq n} a_{ij}^2$.

Proof. Since Δ is the largest degree of $\mathcal{P}(C_n)$

$$|\lambda_1| \leq \Delta \leq n - 1.$$

Using Lemma 2.3 and Theorem 2.5, we have

$$E \geq \frac{2\sqrt{|\lambda_1||\lambda_n|}\sqrt{ns}}{|\lambda_1|+|\lambda_n|} \geq \frac{2\sqrt{n-1}\sqrt{ns}}{n} = 2\sqrt{\frac{n-1}{n}}s,$$

so, the proof is completed.

3. Conclusion

In this study, lower and upper bounds were obtained for the energies of the power graphs of finite cyclic groups by considering the adjacency matrix structure of the power graph of a finite cyclic group. Then, boundary studies were carried out for the energies of the power graphs of finite cyclic groups. Some of the bounds found give results closer to the value. However, it is not always possible to make a boundary comparison as the approaches of the boundaries will change as the graph structure changes.

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