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The Upper Bound For The Largest Signless Laplacian Eigenvalue Of Weighted Graphs

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ABSTRACT

In this study, we find an upper bound for the largest signless Laplacian eigenvalue of simple connected weighted graphs, where edge weights are positive definite square matrices. Also we obtain some results on weighted and unweighted graphs by using this bound.

Key Words: Weighted graph, singles Laplacian eigenvalue, upper bound

1. INTRODUCTION

In this paper, we consider a simple connected weighted graph in which the edge weights are positive definite square matrices. Let be a simple connected weighted graph on vertices. Denote by the positive definite weight matrix of order of the edge and assume that $w_{ij} = w_{ji}$. Let $w_i = \sum_{j:j\sim i} w_{ij}$, for all $i \in V$.

The signless Laplacian matrix $Q(G)$ of a weighted graph G is a block matrix and defined as $Q(G) = (q_{ij})_{nt\times nt}$, where

$$q_{ij} = \begin{cases} w_i & ; \quad \text{if } i = j, \\ w_{ij} & ; \quad \text{if } i \sim j, \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

In the definitions above, the zero denotes the $t\times t$ zero matrix. Thus $Q(G)$ is a square matrix of order nt. Let q_1 denote the largest signless Laplacian eigenvalue of $Q(G)$ and $q_1(w_{ij})$ denote the largest eigenvalue of w_{ij}.

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Upper bounds for the largest signless Laplacian eigenvalue for unweighted graphs have been investigated to great extent in the literature. In Section 2, we give an upper bound for the largest signless Laplacian eigenvalue of weighted graphs. We also characterize graphs for which equality holds in the upper bound.

Lemma 1.

Let A be a Hermitian $n \times n$ matrix with eigenvalues $q_1 \geq q_2 \geq \ldots \geq q_n$, then for any $\overline{x} \in R^n \left(\overline{x} \neq \overline{0}\right)$, $\overline{y} \in R^n \left(\overline{y} \neq \overline{0}\right)$

$$\left|\overline{x}^T A\overline{y}\right| \leq q_1 \sqrt{\overline{x}^T \overline{x}}\sqrt{\overline{y}^T \overline{y}}\ .$$

Equality holds if and only if $\overline{x}$ is an eigenvector of A corresponding to the largest eigenvalue q_1 and $\overline{y} = \alpha\overline{x}$ for some $\alpha \in R$ [5].

Theorem 1.

If A is a Hermitian $n \times n$ matrix with eigenvalues $q_1 \geq q_2 \geq \ldots \geq q_n$, then for any $\overline{x} \in C^n$

$$q_n \overline{x}^T \overline{x} \leq \overline{x}^T A\overline{x} \leq q_1 \overline{x}^T \overline{x}$$

$$q_{\max} = q_1 = \max_{\overline{x} \neq 0} \frac{\overline{x}^T A\overline{x}}{\overline{x}^T \overline{x}} = \max_{\overline{x}^T \overline{x}=1} \overline{x}^T A\overline{x}$$

$$q_{\min} = q_n = \min_{\overline{x} \neq 0} \frac{\overline{x}^T A\overline{x}}{\overline{x}^T \overline{x}} = \min_{\overline{x}^T \overline{x}=1} \overline{x}^T A\overline{x} \text{ [5].}$$

Corollary 1.

Let $A \in M_n$ have eigenvalues $\{q_i\}$. Even if A is not Hermitian, one has the bounds

$$\min_{\overline{x}\neq 0,\, \overline{y}\neq 0} \left|\frac{\overline{x}^T A\overline{y}}{\overline{x}^T \overline{y}}\right| \leq |q_i| \leq \max_{\overline{x}\neq 0,\, \overline{y}\neq 0} \left|\frac{\overline{x}^T A\overline{y}}{\overline{x}^T \overline{y}}\right|,$$

for any $\overline{x} \in R^n \left(\overline{x} \neq \overline{0}\right)$, $\overline{y} \in R^n \left(\overline{y} \neq \overline{0}\right)$ and for $i = 1, 2, \ldots, n$ [8].

Corollary 2.

Let $A, B \in M_n$ are positive definite matrices and $k \in N$. Then, A^k and $A + B$ are also positive definite matrices [5].

2. MAIN RESULTS

In this section we find an upper bound on the largest signless Laplacian eigenvalue of simple connected weighted graphs.

Theorem 2.

Let G be a simple connected weighted graph. Then

$$q_1 \leq \max_{i \in V}\left\{\sqrt{\begin{array}{l} q_1\left(w_i^2\right) + \sum\limits_{k:k\sim i} q_1\left(w_{ik}^2\right) + \sum\limits_{s:s\sim i} q_1\left(w_i w_{is} + w_{is} w_s\right) \\ \qquad + \sum\limits_{\substack{1\leq i,\, r\leq n \\ i\neq r}} \sum\limits_{\substack{k:k\sim i \\ k\sim r}} q_1\left(w_{ik} w_{kr}\right) \end{array}}\right\}. \quad (1)$$

Moreover equality holds in (1) if and only if

(i) G is a bipartite regular graph or a bipartite semiregular graph,

(ii) w_{ij} have a common eigenvector corresponding to the largest eigenvalue $q_1\left(w_{ij}\right)$ for all i, j.

Proof.

Let us consider the matrix $Q^2(G)$. The $(i, j)-$th element of $Q^2(G)$ is

$$\begin{cases} w_i^{\,2} + \sum\limits_{j:j\sim i} w_{ij}^{\,2} & ; \quad \text{if } i = j, \\ w_i w_{ij} + w_{ij} w_j + \sum\limits_{\substack{k:k\sim i \\ k\sim j}} w_{ik} w_{kj} & ; \quad \text{if } i \sim j, \\ \sum\limits_{\substack{k:k\sim i \\ k\sim j}} w_{ik} w_{kj} & ; \quad \text{otherwise.} \end{cases}$$

Let $\overline{x} = \left(x_1^{\,T}, x_2^{\,T}, \ldots, x_n^{\,T}\right)^T$ be an eigenvector corresponding to the eigenvalue $q_1^{\,2}$ of $Q^2(G)$ and x_i be the vector component of $\overline{x}$ such that

$$x_i^{\,T} x_i = \max_{k \in V}\left\{x_k^{\,T} x_k\right\}. \quad (2)$$

Since $\overline{x}$ is nonezero, so is x_i. We have

$$Q^2(G)\overline{x} = q_1^{\,2}\overline{x}\ . \quad (3)$$

From the $i-$th equation of (3), we get

$$q_1{}^2 x_i = w_i{}^2 x_i + \sum_{k:k\sim i} w_{ik}{}^2 x_i + \sum_{s:s\sim i} (w_i w_{is} + w_{is} w_s) x_s + \sum_{\substack{1\le i,r\le n\\ i\ne r}} \sum_{\substack{k:k\sim i\\ k\sim r}} w_{ik} w_{kr} x_r ,$$

i.e.,

$$q_1{}^2 x_i{}^T x_i \le \left| x_i{}^T w_i{}^2 x_i \right| + \sum_{k:k\sim i} \left| x_i{}^T w_{ik}{}^2 x_i \right| + \sum_{s:s\sim i} \left| x_i{}^T (w_i w_{is} + w_{is} w_s) x_s \right| + \sum_{\substack{1\le i,r\le n\\ i\ne r}} \sum_{\substack{k:k\sim i\\ k\sim r}} \left| x_i{}^T w_{ik} w_{kr} x_r \right| . \quad (4)$$

From Corollary 1 and Lemma 1, we have

$$\le q_1\left(w_i{}^2\right) x_i{}^T x_i + \sum_{k:k\sim i} q_1\left(w_{ik}{}^2\right) x_i{}^T x_i + \sum_{s:s\sim i} \left| x_i{}^T (w_i w_{is} + w_{is} w_s) x_s \right| + \sum_{\substack{1\le i,r\le n\\ i\ne r}} \sum_{\substack{k:k\sim i\\ k\sim r}} \left| x_i{}^T w_{ik} w_{kr} x_r \right| . \quad (5)$$

Four cases arise;

1. $w_i w_{is} + w_{is} w_s$ and $w_{ik} w_{kr}$ are Hermitian matrices for all $s, s\sim i$ and for all $k, k\sim i, k\sim r,\ 1\le i,r\le n$,
2. $w_i w_{is} + w_{is} w_s$ is a Hermitian matrix for all $s, s\sim i$ and $w_{ik} w_{kr}$ is not a Hermitian matrix for all $k,\ k\sim i,\ k\sim r,\ 1\le i,r\le n$,
3. $w_{ik} w_{kr}$ is a Hermitian matrix for all $k, k\sim i, k\sim r,\quad 1\le i,r\le n$ and $w_i w_{is} + w_{is} w_s$ is not a Hermitian matrix for all $s, s\sim i$,
4. $w_i w_{is} + w_{is} w_s$ and $w_{ik} w_{kr}$ are not Hermitian matrices for all $s, s\sim i$ and for all $k, k\sim i, k\sim r,\ 1\le i,r\le n$,

Case 1: $w_i w_{is} + w_{is} w_s$ and $w_{ik} w_{kr}$ are Hermitian matrices. From (5), (2) and using Lemma 1 we get

$$q_1 \le \max_{i\in V} \left\{ \sqrt{ \begin{array}{r} q_1\left(w_i^2\right) + \sum_{k:k\sim i} q_1\left(w_{ik}^2\right) + \sum_{s:s\sim i} q_1\left(w_i w_{is} + w_{is} w_s\right) \\ + \sum_{\substack{1\le i,r\le n\\ i\ne r}} \sum_{\substack{k:k\sim i\\ k\sim r}} q_1\left(w_{ik} w_{kr}\right) \end{array} } \right\}$$

Case 2: $w_i w_{is} + w_{is} w_s$ is a Hermitian matrix and $w_{ik} w_{kr}$ is not a Hermitian matrix. Let us take the ratio of

$$\left| \frac{x_\ell{}^T w_{\ell k} w_{kr} x_r}{x_\ell{}^T x_r} \right|$$

for $1\le \ell, r\le n$. If $\left| N_\ell \cap N_r \right| = 0$, this ratio is zero, where $\left| N_\ell \cap N_r \right|$ is the number of common neighbors of ℓ and r. So let us consider $\left| N_\ell \cap N_r \right| \ne 0$. From (2) and using the Cauchy-Schwarz inequality, we have

$$\left| \frac{x_\ell{}^T w_{\ell k} w_{kr} x_r}{x_\ell{}^T x_r} \right| \ge \frac{\left| x_\ell{}^T w_{\ell k} w_{kr} x_r \right|}{\sqrt{x_\ell{}^T x_\ell}\sqrt{x_r{}^T x_r}} \ge \frac{\left| x_\ell{}^T w_{\ell k} w_{kr} x_r \right|}{\sqrt{x_i{}^T x_i}\sqrt{x_r{}^T x_r}} . \quad (6)$$

Since (6) implies for each x_ℓ and x_r

$$\min_{x_\ell\ne 0, x_r\ne 0} \left\{ \frac{x_\ell{}^T w_{\ell k} w_{kr} x_r}{x_\ell{}^T x_r} \right\} = \frac{\left| x_i{}^T w_{ik} w_{kr} x_r \right|}{\sqrt{x_i{}^T x_i}\sqrt{x_r{}^T x_r}} . \quad (7)$$

From (2), (7) and using Corollary 1, we get

$$\sum_{\substack{1\le i,r\le n\\ i\ne r}} \sum_{\substack{k:k\sim i\\ k\sim r}} \left| x_i{}^T w_{ik} w_{kr} x_r \right| \le \sum_{\substack{1\le i,r\le n\\ i\ne r}} \sum_{\substack{k:k\sim i\\ k\sim r}} q_1\left(w_{ik} w_{kr}\right) x_i{}^T x_i . \quad (8)$$

Since $w_i w_{is} + w_{is} w_s$ is a Hermitian matrix, from (2), (5), (8) and using Lemma 1, we get

$$q_1 \le \max_{i\in V} \left\{ \sqrt{ \begin{array}{r} q_1\left(w_i^2\right) + \sum_{k:k\sim i} q_1\left(w_{ik}^2\right) + \sum_{s:s\sim i} q_1\left(w_i w_{is} + w_{is} w_s\right) \\ + \sum_{\substack{1\le i,r\le n\\ i\ne r}} \sum_{\substack{k:k\sim i\\ k\sim r}} q_1\left(w_{ik} w_{kr}\right) \end{array} } \right\} .$$

Case 3: $w_{ik} w_{kr}$ is a Hermitian matrix and $w_i w_{is} + w_{is} w_s$ is not a Hermitian matrix. By a similar argument to Case 2 we have

$$\sum_{s:s\sim i} \left| x_i{}^T (w_i w_{is} + w_{is} w_s) x_s \right| \le \sum_{s:s\sim i} q_1\left(w_i w_{is} + w_{is} w_s\right) x_i{}^T x_i . \quad (9)$$

Since $w_{ik} w_{kr}$ is a Hermitian matrix, from (2), (5), (9) and using Lemma 1, we get

$$q_1 \le \max_{i\in V}\left\{\sqrt{q_1\left(w_i^2\right)+\sum_{k:k\sim i}q_1\left(w_{ik}^2\right)+\sum_{s:s\sim i}q_1\left(w_i w_{is}+w_{is}w_s\right)+\sum_{\substack{1\le i,r\le n\\ i\ne r}}\sum_{\substack{k:k\sim i\\ k\sim r}}q_1\left(w_{ik}w_{kr}\right)}\right\}$$

Case 4: $w_i w_{is} + w_{is}w_s$ and $w_{ik}w_{kr}$ are not Hermitian matrices. By applying the same methods as Case 2 and Case 3, we can show that

$$\begin{aligned}
q_1^2 x_i^T x_i &\le q_1\left(w_i^2\right)x_i^T x_i + \sum_{k:k\sim i}q_1\left(w_{ik}^2\right)x_i^T x_i + \sum_{s:s\sim i}\left|x_i^T\left(w_i w_{is}+w_{is}w_s\right)x_s\right| + \sum_{\substack{1\le i,r\le n\\ i\ne r}}\sum_{\substack{k:k\sim i\\ k\sim r}}\left|x_i^T w_{ik}w_{kr}x_r\right| \\
&\le q_1\left(w_i^2\right)x_i^T x_i + \sum_{k:k\sim i}q_1\left(w_{ik}^2\right)x_i^T x_i + \sum_{s:s\sim i}q_1\left(w_i w_{is}+w_{is}w_s\right)\sqrt{x_i^T x_i}\sqrt{x_s^T x_s} \\
&\quad + \sum_{\substack{1\le i,r\le n\\ i\ne r}}\sum_{\substack{k:k\sim i\\ k\sim r}}q_1\left(w_{ik}w_{kr}\right)\sqrt{x_i^T x_i}\sqrt{x_r^T x_r}.
\end{aligned} \tag{10}$$

From (2), we get

$$q_1 \le \max_{i\in V}\left\{\sqrt{q_1\left(w_i^2\right)+\sum_{k:k\sim i}q_1\left(w_{ik}^2\right)+\sum_{s:s\sim i}q_1\left(w_i w_{is}+w_{is}w_s\right)+\sum_{\substack{1\le i,r\le n\\ i\ne r}}\sum_{\substack{k:k\sim i\\ k\sim r}}q_1\left(w_{ik}w_{kr}\right)}\right\}.$$

Now suppose that equality holds in (1). Then all inequalities in the above argument must be equalities. From equality in (10), we have

$$x_k^T x_k = x_i^T x_i, \tag{11}$$

for all $k, k\sim i$ and for all $k, k\sim p, p\sim i$. From equality in (10) and using Lemma 1, we get

$$x_s = a_{is}x_i \text{ and } x_r = b_{ir}x_i, \tag{12}$$

for any $s,\ s\sim i$ and for any $r, k\sim r, k\sim i,$ $1\le i,r\le n$, where $a_{is}, b_{ir}\in R$. From (11) and (12), we get

$$\left(a_{is}^2-1\right)x_i^T x_i = 0 \text{ and } \left(b_{ir}^2-1\right)x_i^T x_i = 0,$$

i.e.,

$$a_{is}, b_{ir} = \pm 1, \text{as } x_i^T x_i \ge 0. \tag{13}$$

On the other hand from Corollary 2 w_i^2 and w_{ik}^2 are positive definite matrices for a $1\le i,k\le n$. Thus, we get

$$x_i^T w_i^2 x_i > 0 \text{ and } x_i^T w_{ik}^2 x_i > 0 \tag{14}$$

From (4), (13) and (14), we have

$$\begin{aligned}
&\sum_{s:s\sim i}\left(a_{is}x_i^T\left(w_i w_{is}+w_{is}w_s\right)x_i - \left|x_i^T\left(w_i w_{is}+w_{is}w_s\right)x_i\right|\right) \\
&+\sum_{\substack{1\le i,r\le n\\ i\ne r}}\sum_{\substack{k:k\sim i\\ k\sim r}}\left(b_{ir}x_i^T w_{ik}w_{kr}x_i - \left|x_i^T w_{ik}w_{kr}x_i\right|\right) = 0.
\end{aligned} \tag{15}$$

Four cases arise;

i. $x_i^T\left(w_i w_{is}+w_{is}w_s\right)x_i \ge 0$, $x_i^T w_{ik}w_{kr}x_i \ge 0$,

ii. $x_i^T\left(w_i w_{is}+w_{is}w_s\right)x_i \ge 0$, $x_i^T w_{ik}w_{kr}x_i < 0$,

iii. $x_i^T\left(w_i w_{is}+w_{is}w_s\right)x_i < 0$, $x_i^T w_{ik}w_{kr}x_i \ge 0$,

iv. $x_i^T\left(w_i w_{is}+w_{is}w_s\right)x_i < 0$, $x_i^T w_{ik}w_{kr}x_i < 0$.

Case i: $x_i^T\left(w_i w_{is}+w_{is}w_s\right)x_i \ge 0$ and $x_i^T w_{ik}w_{kr}x_i \ge 0$.

From (15), we get $a_{is} = 1$ and $b_{ir} = 1$, for all $s, s\sim i$ and for all $1\le i,r\le n$.

Let $U = \{k : x_k = x_i, k\sim i\}$ and $W = \{k : x_k = x_i,\ k \nsim i\}$. Since $U\cap W = \varnothing$ and $U\cup W = V$, G is bipartite. Now we have

$$Q(G)x_i = q_1 x_i,$$

i.e.,

$$q_1 x_i = w_i x_i + \sum_{k:k\sim i} w_{ik} x_k .$$

For $i, j \in U$,

$$q_1 x_i = q_1(w_i)x_i + \sum_{k:k\sim i} q_1(w_{ik})x_i , \qquad (16)$$

$$q_1 x_i = q_1(w_j)x_i + \sum_{k:k\sim j} q_1(w_{jk})x_i . \qquad (17)$$

From (16), (17) and $x_i \neq 0$, we get $q_1(w_i) = q_1(w_j)$. Therefore $q_1(w_i)$ is constant for all $i \in U$. Similarly we can also show that $q_1(w_i)$ is constant for all $i \in W$. Hence G is a bipartite regular graph.

Case ii:
$x_i^T(w_i w_{is} + w_{is} w_s)x_i \geq 0$ and $x_i^T w_{ik} w_{kr} x_i < 0.$

From (15), we get $a_{is} = 1$ and $b_{ir} = -1$, for all $s, s \sim i$ and for all $1 \leq i, r \leq n$. Let $U = \{k : x_k = x_i\}$ and $W = \{k : x_k = -x_i\}$. Since $U \cap W = \emptyset$ and $U \cup W = V, G$ is bipartite. By a similar argument Case i, we can show that $q_1(w_i)$ is constant for all $i \in U$ and $q_1(w_j)$ is constant for all $j \in W$. Hence G is a bipartite semiregular graph.

Case iii:
$x_i^T(w_i w_{is} + w_{is} w_s)x_i < 0$ and $x_i^T w_{ik} w_{kr} x_i \geq 0$,

Case iv:
$x_i^T(w_i w_{is} + w_{is} w_s)x_i < 0$ and $x_i^T w_{ik} w_{kr} x_i < 0$.

By applying the same methods as Case i and Case ii, we can show that G is a bipartite regular graph or a bipartite semiregular graph.

Coversely, suppose that conditions (i)-(ii) of theorem hold for the graph G.

G is a bipartite regular graph. Let U and W be the vertex classes of G. Also, let $q_1(w_i) = q_1(w_j) = \alpha$ for $i \in U$ and for $j \in W$. The following equation can be easily verified:

$$4\alpha^2 \begin{pmatrix} x \\ \vdots \\ x \end{pmatrix} =$$

$$\begin{pmatrix} w_1^2 + \sum_{k:k\sim 1} w_{1k}^2 & \cdots & w_1 w_{1n} + w_{1n} w_n + \sum_{\substack{k:k\sim 1 \\ k\sim n}} w_{1k} w_{kn} \\ \vdots & \vdots & \vdots \\ w_n w_{n1} + w_{n1} w_1 + \sum_{\substack{k:k\sim n \\ k\sim 1}} w_{nk} w_{k1} & \cdots & w_n^2 + \sum_{k:k\sim n} w_{nk}^2 \end{pmatrix} \begin{pmatrix} x \\ \vdots \\ x \end{pmatrix}.$$

Thus $4\alpha^2$ is an eigenvalue of $Q^2(G)$. So,

$$2\alpha \leq q_1 .$$

On the other hand, we have

$$q_1 \leq \max_{i\in V} \left\{ \sqrt{ q_1(w_i^2) + \sum_{k:k\sim i} q_1(w_{ik}^2) + \sum_{s:s\sim i} q_1(w_i w_{is} + w_{is} w_s) + \sum_{\substack{1\leq i,r\leq n \\ i\neq r}} \sum_{\substack{k:k\sim i \\ k\sim r}} q_1(w_{ik} w_{kr}) } \right\} = 2\alpha .$$

Thus

$$q_1 = \max_{i\in V} \left\{ \sqrt{ q_1(w_i^2) + \sum_{k:k\sim i} q_1(w_{ik}^2) + \sum_{s:s\sim i} q_1(w_i w_{is} + w_{is} w_s) + \sum_{\substack{1\leq i,r\leq n \\ i\neq r}} \sum_{\substack{k:k\sim i \\ k\sim r}} q_1(w_{ik} w_{kr}) } \right\}.$$

Hence the theorem is proved.

Corollary 3.

Let G be a simple connected weighted graph where each edge weight w_{ij} is a positive number. Then

$$q_1 \leq \max_{i\in V} \left\{ \sqrt{ w_i^2 + \sum_{k:k\sim i} w_{ik}^2 + \sum_{s:s\sim i} (w_i w_{is} + w_{is} w_s) + \sum_{\substack{1\leq i,r\leq n \\ i\neq r}} \sum_{\substack{k:k\sim i \\ k\sim r}} w_{ik} w_{kr} } \right\}. \qquad (18)$$

Moreover equality holds in (18) if and only if G is a bipartite regular graph or a bipartite semiregular graph.

Proof.

For weighted graph where the edge weights w_{ij} are positive number, we have

$$q_1(w_{ij}) = w_{ij} \text{ and } q_1(w_i) = w_i ,$$

for all i, j. Using Theorem 2 we get the required result.

Corollary 4.

Let G be a simple connected unweighted graph. Then

$$q_1 \le \max_{i \in V} \left\{ \sqrt{d_i^{\,2} + d_i + \sum_{j:j\sim i} \left(d_i + d_j\right) + \sum_{\substack{1\le i,j\le n \\ i\ne j}} \left|N_i \cap N_j\right|} \right\}, \quad (19)$$

where d_i is the degree of vertex i and $\left|N_i \cap N_j\right|$ is the number of common neighbors of i and j. Moreover, equality holds in (19) if and only if G is a bipartite regular graph or a bipartite semiregular graph.

Proof.

For an unweighted graph, $w_{ij} = 1$ and $w_i = d_i$ for all i, j and $i \sim j$. Using Corollary 3 we get the required result.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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