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Research Article

Case's Method for the Anlı-Güngör Scattering Formula

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Abstract: Case method is a powerful method in solving one-speed neutron transport equation. The method can be applied to one-speed neutron transport problems and pseudo- geometry problems. The method basis on the usage of Case's eigenfunctions and the orthogonality relations with the certain boundary conditions according to the interested problem. The scattering effects can be investigated via Mika scattering formula and also İnönü's scattering formula. In this study Case method's formalism is derived by using Anlı-Güngör scattering formula as an analogue of Mika's scattering function. This study is about the Case's eigenfunctions, normalization relation and the orthogonality properties among these eigenfunctions and, moreover; Case's eigenfunctions and the orthogonality properties must be rewritten according to the studied scattering order as the number of scattering order increase in Anlı-Güngör scattering formula.

Key words: Case's method, Case's eigenfunctions, Mika's scattering function, Anlı-Güngör scattering function

Anlı-Güngör Saçılma Formülü için Case Metodu

Öz: Case metodu, tek-hızlı nötron transport denkleminin çözümünde güçlü bir metottur. Case metodu tek hızlı nötron transport problemlerine ve psedo-geometrilere uygulanabilir. Method, ilgilenen problemin özelliklerine göre belirli sınır şartlarıyla Case özfonksiyonları ve bu özfonksiyonlar arasındaki diklik bağıntılarının kullanımına dayanır. Saçılma etkileri Mika saçılma formülü ve İnönü saçılma formülü ile araştırılabilir. Bu çalışmada Case metodunun formalizmi, Mika saçılma fonksiyonunun analoğu olarak Anlı-Güngör saçılma formülü için türetilmiştir. Bu çalışma, Case özfonksiyonlarını, normalizasyon bağıntısını ve bu özfonksiyonlar arasındaki diklik bağıntıları ile ilgilidir ve dahası Anlı-Güngör saçılma formülündeki saçılma mertebesinin sayısı arttıkça Case özfonksiyonları ve diklik bağıntıları çalışılan saçılma parametresine göre yeniden yazılmalıdır.

Anahtar kelimeler: Case metodu, Case özfonksiyonları, Mika saçılma formülü, Anlı-Güngör saçılma fonksiyonu

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1. Introduction

The solution of one-speed, homogeneous medium and time-independent neutron transport equation is important to get approximate solutions in neutron transport theory. The equation in 3D has seven independent variables. Therefore, some approximations are performed to get a solution with analytical methods. These are reasonable approximations such as homogeneous medium, time-independent and one-speed approximations etc.

The one-speed, time independent and homogeneous medium neutron transport equation is a homogeneous equation in the absence of neutron sources. This homogeneous equation was solved by Case [1,2]. His method basis on the usage of the orthogonality properties of Case eigenfunctions which are discrete and continuum eigenfunctions. The method can be also applied two different orthogonality properties, half-range and full-range. But it is well known that half-space problems can be investigated by using the Placzek lemma [3]. This lemma is used to transform the half-space problems to full-space problems.

The scattering properties in one-speed reactor theory problems are investigated by Mika scattering function [3] and also İnönü's scattering function [4,5]. Mika scattering function is written in terms of Legendre polynomials and scattering coefficients. İnönü scattering function is written by forward and backward scattering. İnönü showed that the solution of the one-speed neutron transport equation with İnönü's scattering can be written by Case's method. The linear anisotropic scattering has been applied to the original spherical geometry problem by Sahni [7].

Recently studied Anlı-Güngör scattering function [8] is based on the usage of the Legendre polynomials too. But Mika anisotropic scattering function can be written by partial scatterings such as pure-linear anisotropic scattering, pure-quadratic anisotropic scattering, linear-quadratic anisotropic scattering, which is the combination of the pure-linear and pure-quadratic anisotropic scatterings, etc. But Anlı-Güngör scattering function cannot be written by partial scatterings. The quadratic scattering function in Anlı-Güngör scattering must include the linear anisotropic scattering. Therefore, to investigate the reactor theory problems with Anlı-Güngör scattering function could be interesting.

In this study, Case's method formalism is improved for Anlı-Güngör scattering function as an analogue of Mika scattering function. The investigation is performed for half-space.

2. Material and Method

2.1 Case method for the Anlı-Güngör scattering

The one-speed, time-independent and homogeneous medium transport equation for free source is given as

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \psi(x,\mu) = \frac{c}{2}\int_{-1}^{1} f(\mu,\mu')\psi(x,\mu')\,d\mu' \tag{1}$$

here x is the spatial variable in mfp unit, μ is the direction cosine, c is the secondary neutron number and $f(\mu,\mu')$ is the scattering function. Mika scattering function is written as

$$f(\mu,\mu')=\sum_{\ell=0}^{N}(2\ell+1)f_\ell P_\ell(\mu)P_\ell(\mu'). \tag{2}$$

Case method was improved by Mika for his scattering function. If $N=0$ with $f_0=1$, then the scattering is defined as isotropic scattering. The scattering probabilities are equal in every direction in isotropic scattering. If $N=1$ with $f_0=1$ and $f_1\in[-1/3,1/3]$, then the scattering is defined as linear anisotropic scattering, etc. The Anlı-Güngör scattering function is given as

$$f(\mu,\mu')=\sum_{n=0}^{N}t^n P_n(\mu)P_n(\mu') \tag{3}$$

where t is the scattering parameter, $|t|\le 1$. If Eq.(2) is used in Eq.(1), then we get

$$\mu\frac{\partial\psi(x,\mu)}{\partial x}+\psi(x,\mu)=\frac{c}{2}\sum_{n=0}^{N}t^n P_n(\mu)\int_{-1}^{1}P_n(\mu')\psi(x,\mu')d\mu'. \tag{4}$$

If the neutron flux is taken as

$$\psi(x,\mu)=\phi(\nu,\mu)e^{-x/\nu} \tag{5}$$

and Eq.(5) is used in Eq.(4), then we get

$$\left(1-\frac{\mu}{\nu}\right)\phi(\nu,\mu)=\frac{c}{2}\sum_{n=0}^{N}t^n P_n(\mu)\underbrace{\int_{-1}^{1}P_n(\mu')\phi(\nu,\mu')d\mu'}_{J_n(\nu)}. \tag{6}$$

where $J_n(\nu)$ corresponds to

$$J_n(\nu)=\int_{-1}^{1}P_n(\mu')\phi(\nu,\mu')d\mu'. \tag{7}$$

Thus Case eigenfunction for the Anlı-Güngör scattering is

$$(\nu-\mu)\phi(\nu,\mu)=\frac{c\nu}{2}\sum_{n=0}^{N}t^n P_n(\mu)J_n(\nu) \tag{8}$$

or we can define Eq.(8) as following

$$\phi(\nu,\mu)=\frac{c\nu}{2}\frac{\sum_{n=0}^{N}t^n P_n(\mu)J_n(\nu)}{\nu-\mu}=\frac{c\nu}{2}\frac{K_n(\nu,\mu)}{\nu-\mu} \tag{9}$$

where

$$K_n(\nu,\mu)=\sum_{n=0}^{N}t^n P_n(\mu)J_n(\nu) \tag{10}$$

$J_n(\nu)$ which is given in Eq.(7) has a recursion relation. If. Eq.(8) is multiplied with $P_k(\nu)$, and both the orthogonality relation, Eq.(11a), and the recursion relation, Eq.(11b), between Legendre polynomials are applied,

$$\int_{-1}^{1} P_n(\mu) P_k(\mu) d\mu = \frac{2}{2k+1}\delta_{n,k} \tag{11a}$$

$$\mu P_k(\mu) = \frac{k+1}{2k+1} P_{k+1}(\mu) + \frac{k}{2k+1} P_{k-1}(\mu) \tag{11b}$$

then we can find this recursion relation of $J_n(\nu)$ as

$$J_{k+1}(\nu) = \frac{\nu}{k+1}\left[(2k+1) - ct^k\right] J_k(\nu) - \frac{k}{k+1} J_{k-1}(\nu). \tag{12}$$

$J_0(\nu)$ corresponds to the normalization of Case's eigenfunction and its value equals to unity:

$$J_0(\nu) = 1. \tag{12a}$$

If Eq.(8) is directly integrated over $\mu \in [-1,1]$, then $J_1(\nu)$ is found as

$$J_1(\nu) = \nu(1-c). \tag{12b}$$

Upper $J_n(\nu)$ functions can be calculated by using Eqs.(11, 11a and 11b):

$$J_2(\nu) = \frac{1}{2}\left(\nu^2(1-c)(3-ct)-1\right) \tag{12c}$$

$$J_3(\nu) = \frac{\nu}{3}\left[\left(5-ct^2\right)\left[\nu^2(1-c)(3-ct)-1\right] - 4(1-c)\right]. \tag{12d}$$

Thus, Case eigenfunctions for the Anlı-Güngör scattering could be written clearly by using $K_n(\nu,\mu)$ which are written for the scattering order.

$$K_0(\nu,\mu) = 1 \tag{13a}$$

$$K_1(\nu,\mu) = 1 + t\nu(1-c)\mu \tag{13b}$$

$$K_2(\nu,\mu) = 1 + t\nu(1-c)\mu + \frac{t^2}{4}\left(3\mu^2-1\right)\left(\nu^2(1-c)(3-ct)-1\right) \tag{13c}$$

2.2 Normalization

Case eigenfunctions are normalized for the Anlı-Güngör scattering as in Mika scattering.

$$\int_{-1}^{1} \phi(\nu,\mu) d\mu = 1 \tag{14}$$

If. Eq.(9) is used in Eq.(14), then the normalization is found as

$$\frac{c\nu}{2}\int_{-1}^{1}\frac{K_n(\nu,\mu)}{\nu-\mu}d\mu=\frac{c\nu}{2}\sum_n t^n J_n(\nu)\int_{-1}^{1}\frac{P_n(\mu)}{\nu-\mu}=1 \tag{15}$$

If Neumann's formula [9,10] is used in Eq.(15),

$$\int_{-1}^{1}\frac{P_n(\nu)}{\nu-\mu}=2Q_n(\nu) \tag{16}$$

where $Q_n(\nu)$ corresponds to second kind Legendre functions, then a more compact relation can be written for the normalization as following:

$$c\nu\sum_{n=0}^{N}t^n J_n(\nu)Q_n(\nu)=1. \tag{17}$$

If $N=0$ in Eq.(17), then

$$Q_0(\nu)=\frac{1}{2}\ln\left(\frac{\nu+1}{\nu-1}\right) \tag{18a}$$

and the result of the normalization is

$$\frac{c\nu}{2}\ln\left(\frac{\nu+1}{\nu-1}\right)=1. \tag{18b}$$

Eq.(18b) is identical to the isotropic scattering in Mika scattering. If $N=1$ in Eq.(17), then

$$Q_1(\nu)=\frac{1}{2}\nu\ln\left(\frac{\nu+1}{\nu-1}\right)-1 \tag{19a}$$

and the result of normalization is

$$\frac{c\nu}{2}\ln\left(\frac{\nu+1}{\nu-1}\right)=\frac{1+ct\nu^2(1-c)}{1+t\nu^2(1-c)} \tag{19b}$$

Eq.(19b) is identical to the linear anisotropic scattering if $t=3f_1$. Similarly, if $N=2$ in Eq.(17), then the result of the normalization integral is the following

$$c\nu\left[Q_0(\nu)+tJ_1(\nu)Q_1(\nu)+t^2J_2(\nu)Q_2(\nu)\right]=1. \tag{19c}$$

Eqs.(18b, 19b and 19c) which are written for different scattering situations, are transcendental equations. Therefore, these equations for different scattering situations can be solved as numerical methods such as Newton-Raphson method or Muller's method. The numerical solution for $\nu\notin[-1,1]$ gives the discrete eigenvalues which are represented as $\pm\nu_0$. If $\nu\in[-1,1]$, then the integral has a singular point for $\nu=\mu$. The continuum eigenfunction correspond to this situation. Finally, discrete and continuum Case eigenfunctions are given as following, respectively:

$$\phi(\pm\nu_0,\mu)=\frac{\pm c\nu_0}{2}\frac{K_n(\pm\nu_0,\mu)}{\nu_0\mp\mu} \tag{20}$$

$$\phi(\nu,\mu) = \frac{c\nu_0}{2} P \frac{K_n(\nu,\mu)}{\nu_0 - \mu} + \lambda(\nu)\delta(\nu-\mu) \tag{21}$$

where P symbol correspond to Cauchy principal value and $\lambda(\nu)$ is defined as

$$\lambda(\nu) = 1 - \frac{c\nu}{2} P \int_{-1}^{1} \frac{K_n(\nu,\mu)}{\nu - \mu} d\mu \tag{22}$$

Thus the general solution of Eq.(4) becomes

$$\psi(x,\mu) = a_{0+}\phi(\nu_0,\mu)e^{-x/\nu_0} + a_{0-}\phi(-\nu_0,\mu)e^{x/\nu_0} + \int_{-1}^{1} A(\nu)\phi(\nu,\mu)e^{-x/\nu} d\nu \tag{23}$$

2.3 Orthogonality relations

It is well known that Case eigenfunctions have orthogonality relations among them. These orthogonality relations are given as

$$\int_{-1}^{1} \mu\phi(\pm\nu_0,\mu)\phi(\pm\nu_0,\mu) d\mu = M(\pm\nu_0), \quad M(\pm\nu_0) = -M(\mp\nu_0), \tag{24a}$$

$$\int_{-1}^{1} \mu\phi(\pm\nu_0,\mu)\phi(\mp\nu_0,\mu) d\mu = 0, \tag{24b}$$

$$\int_{-1}^{1} \mu\phi(\nu,\mu)\phi(\nu,\mu) d\mu = M(\nu) \tag{24c}$$

The results of the normalization integrals are the same in the Mika scattering situation for $N=0$ and $N=1$, with $t = 3f_1$ in Mika’s scattering.

Isotropic scattering:

$$M(\nu_0) = \frac{c\nu_0^3}{2}\left[\frac{c}{\nu_0^2 - 1} - \frac{1}{\nu_0^2}\right] \tag{25a}$$

Isotropic + linear anisotropic scattering:

$$M(\nu_0) = \frac{c^2\nu_0^3}{2}\left[\frac{1 + 2w\nu_0^2 + w^2\nu_0^4}{\nu_0^2 - 1} - \frac{(1 + wc\nu_0^2)(1 + 3w\nu_0^2)}{\nu_0^2} + 2w\nu_0^2(1 + w\nu_0^2)\right] \tag{25b}$$

where $w = t(1-c)$. The orthogonality relation for the continuum eigenfunction is

$$M(\nu) = \nu\lambda^2(\nu) + \frac{\pi^2 c^2 \nu^3}{4} K_n^2(\nu,\nu) \tag{25c}$$

where the second term comes from Poincaré-Bertrand formula [10] and $\lambda(\nu)$ will change in the terms of the scattering situation.

3 Conclusion and Comments

The solution of one-speed neutron transport is important to get approximate results in reactor theory. There are a lot of numerical and analytical methods have been improved by researchers. Therefore, the addition of the anisotropic scatterings could be important in the reactor theory problems.

Case method relations are investigated in this study. Basically, the solution of Eq.(1) is the same for the first order scattering terms for Mika scattering function and the Anlı-Güngör scattering function, which corresponds to isotropic scattering. The second order scattering parameter in the Anlı-Güngör scattering formula corresponds to $t \equiv 3f_1$. We can call it as linear anisotropic scattering.

The one-speed neutron transport problems have been investigated for certain scattering or scatterings such as linear anisotropic, pure-quadratic, linear-quadratic anisotropic scattering or pure-triplet anisotropic scattering with Mika anisotropic scattering formula. But this separation is not impossible in the Anlı-Güngör scattering formula and the differences from Mika scattering formula appear in the further scatterings for $N \geq 2$. Therefore, if we accept that $t \equiv 3f_1$, then the third scattering term in the Anlı-Güngör scattering should be $t^2 = (3f_1)^2$ etc. Therefore, this t parameter lies in $t \in [-1,1]$. It is obviously that the further scatterings is proportional of $3f_1$, such as $t^2 = (3f_1)^2$, $t^3 = (3f_1)^3$, etc. Moreover, if any researcher deal with the third scattering term, the linear anisotropic scattering situation is automatically is in the scattering, and so on. Equations (16, 20,21,24a and 24c) must be written the order of scattering and the numerical calculations must be calculated with these written mathematical relations.

There is also an important point that the physical meaning of t parameter must be investigated for further scatterings. The definition interval of t is $t \in [-1,1]$ for linear anisotropic scattering in the Anlı-Güngör scattering formula. But the value interval of t for example for quadratic anisotropic scattering is not the same this interval.

It is well known that the dominant scattering in Mika's scattering function is the linear anisotropic scattering. The other scattering terms gives very small effects in any investigated problem. Therefore, the usage of the Anlı-Güngör scattering formula in neutron transport problems could give interesting results since t parameter equals to $3f_1$

Author Statement

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Conflict of Interest

As the authors of this study, we declare that we do not have any conflict of interest statement.

Ethics Committee Approval and Informed Consent

As the authors of this study, we declare that we do not have any ethics committee approval and/or informed consent statement.

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