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Research Article

A new approach to fuzzy partial metric spaces

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Abstract

In this study, we aim to introduce the notion of fuzzy partial metric spaces which is a generalization of crisp partial metric spaces in the fuzzifying view with the distance between ordinary points. For this aim, we first present the concept of fuzzy partial metric spaces by considering the distance as non-negative, upper semi-continuous, normal and convex fuzzy numbers by giving examples. We obtain some useful inequalities under some restrictions in fuzzy partial metric spaces. Then we discuss the relationships with the other metric structures and we point out Banach's fixed point theorem as an application of the proposed properties and relations. Finally, we show that fuzzy partial metric spaces induce some α-level topology, Lowen fuzzy topology, and fuzzifying topology.

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1. Introduction

Since the fuzzy set theory was introduced by Zadeh [33] in 1965, this theory gave a new perspective in a lot of areas of science by allowing us to apply fuzzy behavior to model real situations. In fuzzy set theory, one of the most interesting and considerable research topics is the structure of fuzzy metric spaces and their possible application to several areas. The first approach to this structure was carried out by Menger [21] who considered the distances between points by distribution functions. This structure is also a generalization of crisp metric spaces. Then Schweizer and Sklar [26] initiated the notion of probabilistic metric spaces by using arbitrary triangular functions. After this inspiration, Kramosil and Michalek [18] generalized the concept of probabilistic metric spaces to the fuzzy aspect, named KM-fuzzy metric spaces, and studied the topological view of this notion. Later, George and Veeramani [9] slightly modified the notion of KM-fuzzy metric spaces (named GV-fuzzy metric spaces) to obtain Hausdorff topology and also they carried various well-known theorems in crisp metric spaces to the fuzzy metric spaces. On the other hand, Kaleva and Seikkala [17] approached the structure of fuzzy metric spaces (named KS-fuzzy metric space) as a generalization of probabilistic metric spaces by taking distance between two points to be a non-negative, upper semi-continuous, normal and convex fuzzy numbers. In [25], Roldán et al. gave the interrelationships between fuzzy metric structures

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in detail. After these notions were defined, lots of researchers have continued to relate to fuzzy metric structures [4,5,11,12,15,22,31].

In literature, there are various generalizations of crisp metric spaces by relaxing the axioms (see [7,8,13]). One of this generalizations is the notion of partial metric spaces which is an extension of crisp metric spaces in which the self-distance is not necessarily equal to zero and was given by Matthews [20]. This structure was originally motivated by the experience of computer science, as discussed in [6], they authors showed how the mathematics of nonzero self-distance for metric spaces has been established and is now leading to interesting research into the different aspects. Then, different kinds of fixed point theorems were presented by researchers [2,24,28–30].

In recent years, some authors tried to merge the structures of partial metric and fuzzy metric into a single one with different kinds of views. The first approach was given by Yue and Gu [32] as fuzzy partial metric spaces by using the minimum t-norm and considering the KM-fuzzy metric axioms. The other one is the concept of partial fuzzy metric spaces was given by Sedghi et al. [27] who generalizes the structure of strong GV-fuzzy metric spaces. Some fundamental fixed point theorems and topological properties can be found in [1,3]. Another approach to partial metric space in the fuzzy settings by using the residuum operator was given by Gregori et al. [10].

The aim of this work is to initiate the notion of fuzzy partial metric spaces (in the sense of KS-fuzzy metric spaces) which is a generalization of crisp partial metric spaces in the fuzzifying view with the distance between ordinary points. We obtain some useful inequalities under some restrictions of operators used in triangular inequalities in fuzzy partial metric spaces. Then we discuss the relationships with (quasi-)fuzzy metric structures and we point out Banach's fixed point theorem as an application of the proposed properties and relations. Finally, we show that fuzzy partial metric spaces induce some α-level topology, Lowen fuzzy topology, and fuzzifying topology.

2. Preliminaries

We begin by recalling the necessary notions which are used in the sequel of this paper.

Definition 2.1. [17,23] (1) A fuzzy number is a mapping such that $x : \mathbb{R} \to [0,1]$.

(2) A fuzzy number x is called convex if $x(t_1) \geq min(x(t_2), x(t_3))$ when $t_2 \leq t_1 \leq t_3$.

(3) A fuzzy number x is called normal if there exist a $t_0 \in \mathbb{R}$ such that $x(t_0) = 1$.

(4) An α-level set of x is defined by the set $\{t|x(t) \geq \alpha\}$ where $\alpha \in (0,1]$ and denoted by $[x]_\alpha$. $[x]_\alpha$ is a closed interval such as $[a^\alpha, b^\alpha]$ where $a^\alpha = -\infty$ and $b^\alpha = \infty$ are also admissible.

(5) A fuzzy number x is said to be nonnegative if $x(t) = 0$ for all $t < 0$.

We will denote the set of all upper semi-continuous, convex and normal fuzzy numbers by E and the set of all nonnegative elements of E by G.

Since each real number $x \in \mathbb{R}$ can be taken as a fuzzy number $\bar{x}$ defined as

$$\bar{x}(t) = \begin{cases} 0, & t \neq x \\ 1, & t = x \end{cases},$$

the set of real numbers $\mathbb{R}$ can be embedded in E.

Definition 2.2. [17,23] The algebraic operations on $E \times E$ are defined as follows: for all $x, y \in E$ and $t \in \mathbb{R}$,

(i) $(x + y)(t) = sup_{s\in\mathbb{R}} min(x(s), y(t - s))$,

(ii) $(x - y)(t) = sup_{s\in\mathbb{R}} min(x(s), y(s - t))$,

(iii) $(x.y)(t) = sup_{\substack{s\in\mathbb{R},\\ s\neq 0}} min(x(s), y(t/s))$,

(iv) $(x/y)(t) = sup_{s\in\mathbb{R}} min(x(ts), y(s))$.

The additive and multiplicative identities in E are $\bar{0}$ and $\bar{1}$, respectively. $-x$ is defined as $\bar{0} - x$ and it follows that $(-x)(t) = x(-t)$ for all $t \in \mathbb{R}$ and $x - y = x + (-y)$ for all $x, y \in E$. The absolute value of $x \in E$ is denoted by $|x|$ and defined as

$$|x|(t) = \begin{cases} max(x(t), x(-t)), & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

The following lemma gives the characterizations of algebraic operations on $E \times E$ by α-level sets.

Lemma 2.3 ([17,23]). *Let $x, y \in E$ and $[x]_\alpha = [a_1^\alpha, b_1^\alpha]$, $[y]_\alpha = [a_2^\alpha, b_2^\alpha]$ for all $\alpha \in (0,1]$. Then the following properties hold:*
(i) $[x+y]_\alpha = [a_1^\alpha + a_2^\alpha, b_1^\alpha + b_2^\alpha]$,
(ii) $[x-y]_\alpha = [a_1^\alpha - b_2^\alpha, b_1^\alpha - a_2^\alpha]$,
(iii) $[x.y]_\alpha = [a_1^\alpha . a_2^\alpha, b_1^\alpha . b_2^\alpha]$,
(iv) $[\bar{1}/x]_\alpha = [\frac{1}{b_1^\alpha}, \frac{1}{a_1^\alpha}]$ *(if $a_1^\alpha > 0$),*
(v) $[|x|]_\alpha = [max(0, a_1^\alpha, -b_1^\alpha), max(|a_1^\alpha|, |b_1^\alpha|)]$.

Lemma 2.4 ([17,23]). *Let $[x]_\alpha = [a_1^\alpha, b_1^\alpha]$ and $[y]_\alpha = [a_2^\alpha, b_2^\alpha]$ whenever $x, y \in E$. Then the ordering $\preccurlyeq$ in E defined by*

$$x \preccurlyeq y \Leftrightarrow a_1^\alpha \leq a_2^\alpha \text{ and } b_1^\alpha \leq b_2^\alpha$$

for all $\alpha \in (0,1]$, is a partial ordering.

In [17], the authors define the notion of KS-fuzzy metric spaces (fuzzy metric space, for short) by considering that the distance between two points is a nonnegative, normal, convex and upper semi-continuous fuzzy number as follows:

Definition 2.5 ([16,17]). Let X be a non-empty set and $L, R : [0,1] \times [0,1] \to [0,1]$ be two symmetric, non-decreasing mappings such that $L(0,0) = 0$ and $R(1,1) = 1$. A mapping $d : X \times X \to G$ is called a fuzzy metric on X if the following properties hold for all $x, y \in X$ and $\alpha \in (0,1]$,
(FM1) $d(x,y) = \bar{0}$ iff $x = y$,
(FM2) $d(x,y) = d(y,x)$ for all $x, y \in X$,
(FM3) For all $x, y, z \in X$
(i) $d(x,y)(s+t) \geq L(d(x,z)(s), d(z,y)(t))$ whenever $s \leq \mu_1(x,z)$, $t \leq \mu_1(z,y)$ and $s + t \leq \mu_1(x,y)$,
(ii) $d(x,y)(s+t) \leq R(d(x,z)(s), d(z,y)(t))$ whenever $s \geq \mu_1(x,z)$, $t \geq \mu_1(z,y)$ and $s + t \geq \mu_1(x,y)$
where $[d(x,y)]_\alpha = [\mu_\alpha(x,y), \nu_\alpha(x,y)]$. The quadruple (X, d, L, R) is called a fuzzy metric space.

The value $d(x,y)(t)$ can be thought as the possibility that the distance between x and y is t. Also, the family of the sets $N_x(\varepsilon, \alpha) = \{y \in X | \nu_\alpha(x,y) < \varepsilon\}$ is a basis for a metrizable Hausdorff topology T_d on X and this topology is called the topology generated by the fuzzy metric d. Any crisp topological space (X, T) is called to admit a compatible fuzzy metric if there exists a fuzzy metric space (X, d, L, R) such that $T = T_d$. We also note that a crisp topological space (X, T) is metrizable if and only if it admits a compatible fuzzy metric.

Similar to the usual metric space, we can define the notion of fuzzy quasi metric spaces as follows:

Definition 2.6. Let $L, R : [0,1] \times [0,1] \to [0,1]$ be two mappings satisfies the condition in Definition 2.5 and $q : X \times X \to G$ satisfy the condition (FM3) and the following condition
(FM1*) $q(x,y) = q(y,z) = \bar{0}$ iff $x = y$.
Then q is said to be a fuzzy quasi-metric on X and the quadruple (X, q, L, R) denotes the fuzzy quasi metric space.

Lemma 2.7. *If (X, d, L, R) is a fuzzy (quasi-) metric spaces, then the following assertions are equivalent:*

(i) $L = min$ and $R = max$.

(ii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Definition 2.8 ([17])**.** Let (x_n) be a sequence in a fuzzy metric space (X, d, L, R).

(i) (x_n) is called to be convergent to $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = \bar{0}$.

(ii) (x_n) is called to be a Cauchy sequence if $\lim_{n,m\to\infty} d(x_n, x_m) = \bar{0}$.

(iii) (X, d, L, R) is called to be complete if every Cauchy sequence in X is convergent to some point $x \in X$.

Lemma 2.9 ([19])**.** *Let (x_n) be a sequence in a fuzzy metric space (X, d, L, R). Then the followings are hold for all $\alpha \in (0, 1]$:*

(i) $\lim_{n\to\infty} d(x_n, x) = \bar{0}$ if and only if $\lim_{n\to\infty} \nu_\alpha(x_n, x) = 0$.

(ii) $\lim_{n,m\to\infty} d(x_n, x_m) = \bar{0}$ if and only if $\lim_{n,m\to\infty} \nu_\alpha(x_n, x_m) = 0$.

3. Fuzzy partial metric spaces

In this section, we introduce the concept of fuzzy partial metric spaces which is a generalization of both partial metric spaces and KS-fuzzy metric spaces. We discuss some level forms of triangular inequalities under some restrictions and also we give the definitions of convergence, Cauchy sequence and completeness.

Definition 3.1. Let X be a non-empty set and $L, R : [0, 1] \times [0, 1] \to [0, 1]$ be two symmetric, non-decreasing mappings such that $L(a, b) \leq a$, $L(a, b) \leq b$, $R(a, b) \geq a$ and $R(a, b) \geq b$ for all $a, b \in [0, 1]$. A mapping $p : X \times X \to G$ is called a fuzzy partial metric on X if the following properties hold for all $x, y \in X$ and $\alpha \in (0, 1]$,

(FPM1) $p(x, y) = p(x, x) = p(y, y)$ iff $x = y$,

(FPM2) $p(x, y) = p(y, x)$ for all $x, y \in X$,

(FPM3) $p(x, x) \preccurlyeq p(x, y)$,

(FPM4) For all $x, y, z \in X$

(i) $L(p(x, y)(s + t - u), p(z, z)(u)) \geq L(p(x, z)(s), p(z, y)(t))$ whenever $s \leq \mu_1(x, z)$, $t \leq \mu_1(z, y)$, $u \leq \mu_1(z, z)$ and $s + t - u \leq \mu_1(x, y)$,

(ii) $R(p(x, y)(s + t - u), p(z, z)(u)) \leq R(p(x, z)(s), p(z, y)(t))$ whenever $s \geq \mu_1(x, z)$, $t \geq \mu_1(z, y)$, $u \geq \mu_1(z, z)$ and $s + t - u \geq \mu_1(x, y)$

where $[p(x, y)]_\alpha = [\mu_\alpha(x, y), \nu_\alpha(x, y)]$. The quadruple (X, p, L, R) is called a fuzzy partial metric space.

Example 3.2. Let $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \overline{\mathbb{R}^+}$ be defined by $p(x, y) = \overline{max(x, y)}$ where $\overline{\mathbb{R}^+}$ is the set of $\overline{x}$ for $x \in \mathbb{R}^+$. Then $(\mathbb{R}^+, p, min, max)$ is a fuzzy partial metric space.

Remark 3.3. (1) Each fuzzy metric space (X, d, L, R) is a fuzzy partial metric space.

(2) Any crisp partial metric space is a special case of the fuzzy partial metric space. In fact, if (X, P) is a partial metric space, then $P : X \times X \to \mathbb{R}^+ \subseteq G$ since every non-negative real numbers belongs to the set G. (X, P, L, R) is a fuzzy partial metric space with the choice of $L(a, b) = 0$ and $R(a, b) = 1$.

Lemma 3.4. *(FPM4)-(ii) with $R = max$ is equivalent to the following triangular inequality*

$$\nu_\alpha(x, y) \leq \nu_\alpha(x, z) + \nu_\alpha(z, y) - \nu_\alpha(z, z) \tag{3.1}$$

for all $x, y, z \in X$ and $\alpha \in (0, 1]$.

Proof. Suppose that the triangular inequality (3.1) is satisfied for all $x, y, z \in X$ and $\alpha \in (0, 1]$. Take $s \geq \mu_1(x, z)$, $t \geq \mu_1(z, y)$, $u \geq \mu_1(z, z)$ and $\alpha = p(x, y)(s + t - u)$. Then,

$$s + t - u \leq \nu_\alpha(x, y) \leq \nu_\alpha(x, z) + \nu_\alpha(z, y) - \nu_\alpha(z, z)$$

which implies that $s \leq \nu_\alpha(x,z)$ or $t \leq \nu_\alpha(z,y)$ or $-u \leq -\nu_\alpha(z,z)$. Thus, we have that $p(x,z)(s) \geq \alpha$ or $p(z,y)(t) \geq \alpha$ or $p(z,z)(u) < \alpha$. Hence,

$$max(p(x,z)(s), p(z,y)(t)) \geq \alpha = p(x,y)(s+t-u) = max(p(x,y)(s+t-u), p(z,z)(u)).$$

Now, suppose that (FPM4)(ii) with $R = max$ is satisfied and let $x, y, z \in X$ and $\alpha \in (0,1]$. One can assume that $\nu_\alpha(x,z) < \infty$ and $\nu_\alpha(z,y) < \infty$. Otherwise (3.1) is satisfied directly. Suppose that

$$\nu_\alpha(x,y) > \nu_\alpha(x,z) + \nu_\alpha(z,y) - \nu_\alpha(z,z).$$

Take $u = \mu_\alpha(z,z)$. Then there exist $s > \nu_\alpha(x,z) \geq \mu_1(x,z)$ and $t > \nu_\alpha < (z,y) \geq \mu_1(z,y)$ such that $s+t-u = \nu_\alpha(x,y) \geq \mu_1(x,y)$. By (FPM4)(ii), we have that

$$\begin{aligned} \alpha = p(x,y)(s+t-u) &\leq max(p(x,y)(s+t-u), p(z,z)(u)) \\ &\leq max(p(x,z)(s), p(z,y)(t)) < \alpha \end{aligned}$$

which means a contradiction. Hence assumption is not true and the triangular inequality (3.1) is satisfied for all $x, y, z \in X$ and $\alpha \in (0,1]$. □

Lemma 3.5. *(FPM4)-(i) with $L = min$ is equivalent to the following triangular inequality*

$$\mu_\alpha(x,y) \leq \mu_\alpha(x,z) + \mu_\alpha(z,y) - \mu_\alpha(z,z) \tag{3.2}$$

for all $x, y, z \in X$ and $\alpha \in (0,1]$.

Proof. Suppose that the triangular inequality (3.2) is satisfied for all $x, y, z \in X$ and $\alpha \in (0,1]$. Let $s \leq \mu_1(x,z)$, $t \leq \mu_1(z,y)$, $u \leq \mu_1(z,z)$ and $s+t-u \leq \mu_1(x,y)$. Take $\alpha = p(x,z)(s)$, $\beta = p(z,y)(t)$, $\varepsilon = min(\alpha,\beta)$ and $u = \mu_\varepsilon(z,z)$. From here, we have $\mu_\alpha(x,z) \leq s$ and $\mu_\alpha(z,y) \leq t$. From the triangular inequality (3.2), we have

$$\mu_\varepsilon(x,y) \leq \mu_\varepsilon(x,z) + \mu_\varepsilon(z,y) - \mu_\varepsilon(z,z).$$

Since μ_α is non-decreasing with respect to α, we obtain

$$\mu_\varepsilon(x,y) + \mu_\varepsilon(z,z) \leq \mu_\varepsilon(x,z) + \mu_\varepsilon(z,y) \leq \mu_\alpha(x,z) + \mu_\alpha(z,y) \leq s+t.$$

Thus, we have $\mu_\varepsilon(x,y) \leq s+t-u$ which means that $p(x,y)(s+t-u) \geq \varepsilon$. Consequently, the following inequality is obtained

$$min(p(x,y)(s+t-u), p(z,z)(u)) \geq \varepsilon = min(p(x,z)(s), p(z,y)(t)).$$

as desired. Now, suppose that (FPM4)(i) with $L = min$ is satisfied and let $x, y, z \in X$ and $\alpha \in (0,1]$. If $\mu_\alpha(x,z) + \mu_\alpha(z,y) - \mu_\alpha(z,z) \leq \mu_1(x,y)$, then by (FPM4)(i)

$$\begin{aligned} & min(p(x,y)(\mu_\alpha(x,z) + \mu_\alpha(z,y) - \mu_\alpha(z,z)), p(z,z)(\mu_\alpha(z,z))) \\ & \geq min(p(x,z)(\mu_\alpha(x,z)), p(z,y)(\mu_\alpha(z,y))) \geq \alpha, \end{aligned}$$

This means that $p(x,y)(\mu_\alpha(x,z) + \mu_\alpha(z,y) - \mu_\alpha(z,z)) \geq \alpha$. Hence, we obtain that $\mu_\alpha(x,z) + \mu_\alpha(z,y) - \mu_\alpha(z,z) \geq \mu_\alpha(x,y)$. If $\mu_\alpha(x,z) + \mu_\alpha(z,y) - \mu_\alpha(z,z) \geq \mu_1(x,y)$, then we have $\mu_1(x,y) \geq \mu_\alpha(x,y)$ since μ_α is non-decreasing with respect to α. So, the proof is completed. □

Corollary 3.6. *If (X, p, L, R) is a fuzzy partial metric space, then the following assertions are equivalent:*

(i) $L = min$ and $R = max$.

(ii) $p(x,y) + p(z,z) \preccurlyeq p(x,z) + p(z,y)$ for all $x, y, z \in X$.

Corollary 3.7. *If (X, P) is a crisp partial metric space, then (X, p, min, max) is a fuzzy partial metric space where $p(x,y)(t) = \bar{0}(t - P(x,y))$ for all $t \geq 0$.*

Example 3.8. Let $X = E$ and define $p : X \times X \to G$ by $p(x, y) = |x - y|$ for all $x, y \in X$. If $[x]_\alpha = [a_1^\alpha, b_1^\alpha]$, $[y]_\alpha = [a_2^\alpha, b_2^\alpha]$ and $[z]_\alpha = [a_3^\alpha, b_3^\alpha]$, then

$$[p(x, y)]_\alpha = [max(0, a_1^\alpha - b_2^\alpha, a_2^\alpha - b_1^\alpha), max(|a_1^\alpha - b_2^\alpha|, |a_2^\alpha - b_1^\alpha|)]$$

$$[p(z, z)]_\alpha = [\overline{0}]_\alpha = \{0\}.$$

It is easily seen that triangular inequality (3.1) is satisfied with the choice of $L = 0$ and $R = max$. Thus $(X, p, 0, max)$ is a fuzzy partial metric space.

Definition 3.9. Let (X, p, L, R) be a fuzzy partial metric space, (x_n) be a sequence in X and $x \in X$.

(i) (x_n) is said to converge to x if $lim_{n\to\infty} p(x_n, x) = p(x, x)$.

(ii) (x_n) is said to be a Cauchy sequence if $lim_{n,m\to\infty} p(x_n, x_m)$ exists.

(iii) (X, p, L, R) is said to be complete if each Cauchy sequence is convergent to a point of $x \in X$ such that $lim_{n\to\infty} p(x_n, x) = lim_{n,m\to\infty} p(x_n, x_m) = p(x, x)$.

Remark 3.10. In fuzzy partial metric spaces, each convergent sequence may not be a Cauchy sequence, and each convergent sequence may not be a unique limit point.

Definition 3.11. Let (X, p, L, R) be a fuzzy partial metric space, (x_n) be a sequence in X and $x \in X$. (x_n) is said to p-converge to x if $lim_{n\to\infty} p(x_n, x) = lim_{n\to\infty} p(x_n, x_n) = p(x, x)$.

Lemma 3.12. *In fuzzy partial metric space (X, p, min, max), each p-convergent sequence is a Cauchy sequence.*

Definition 3.13. Let (X, p, L, R) be a fuzzy partial metric space and (x_n) be a sequence in X. (x_n) is said to be a 0-Cauchy sequence if $lim_{n,m\to\infty} p(x_n, x_m) = \overline{0}$.

Definition 3.14. (X, p, L, R) is said to be 0-complete if each 0-Cauchy sequence is convergent to a point of $x \in X$ such that $p(x, x) = \overline{0}$.

Lemma 3.15. *Each complete fuzzy partial metric spaces is a 0-complete fuzzy partial metric space.*

Example 3.16. Let us consider the fuzzy partial metric space (X, p, min, max) given in Example 3.2. It is straightforward to check that (X, p, min, max) is a 0-complete space that is not complete.

4. The relations between fuzzy partial metric spaces and fuzzy (quasi-) metric spaces

In this section, we show the relations with fuzzy (quasi-) metric spaces and fuzzy partial metric spaces and topologies induced by these metrics.

Theorem 4.1. *Let (X, p, min, max) be a fuzzy partial metric space. Then the mapping $q_p : X \times X \to G$ defined by*

$$q_p(x, y) = p(x, y) - p(x, x)$$

is a fuzzy quasi metric on X.

Proof. (FM1*) Let $q_p(x, y) = q_p(y, x) = \bar{0}$. Then we have $p(x, y) = p(x, x) = p(y, y)$ from definition of q_p. Since p is a fuzzy partial metric on X, then $x = y$. It is clear that $q_p(x, y) = q_p(y, x) = \bar{0}$ when $x = y$.

(FM3) By Corollary 3.6, we have

$$q_p(x, y) = p(x, y) - p(x, x) \preccurlyeq p(x, z) + p(z, y) - p(z, z) - p(x, x) = q_p(x, z) + q_p(z, y).$$

From Lemma 3.6, the condition (FM3) is satisfied. Hence, (X, q_p, min, max) is a fuzzy quasi metric space. $\square$

Theorem 4.2. *Let (X, p, L, R) be a fuzzy partial metric space and define the mapping $d_p : X \times X \to G$ as*

$$d_p(x, y) = \begin{cases} p(x, y), & x \neq y \\ \overline{0}, & x = y \end{cases}$$

for all $x, y \in X$. Then (X, d_p, L, R) is a fuzzy metric space and also, note that (X, p, L, R) is 0-complete if and only if (X, d_p, L, R) is complete.

Proof. It is clear from the definition of d_p that $d_p(x, y) = \overline{0}$ iff $x = y$ and $d_p(x, y) = d_p(y, x)$ for all $x, y \in X$. Let $x, y, z \in X$.

(i) From (FPM4)(i), we know that

$$L(p(x, y)(s + t - u), p(z, z)(u)) \geq L(p(x, z)(s), p(z, y)(t))$$

whenever $s \leq \mu_1(x, z)$, $t \leq \mu_1(z, y)$, $u \leq \mu_1(z, z)$ and $s + t - u \leq \mu_1(x, y)$. From here, we obtain

$$p(x, y)(s + t - u) \geq L(p(x, y)(s + t - u), p(z, z)(u)) \geq L(p(x, z)(s), p(z, y)(t))$$

whenever $s \leq \mu_1(x, z)$, $t \leq \mu_1(z, y)$, $u \leq \mu_1(z, z)$ and $s+t-u \leq \mu_1(x, y)$. Since $p(x, y) \in G$ is non-decreasing on $(0, \mu_1(x, y)]$, we have that

$$\begin{aligned} d_p(x, y)(s + t) &= p(x, y)(s + t) \geq p(x, y)(s + t - u) \geq L(p(x, y)(s + t - u), p(z, z)(u)) \\ &\geq L(p(x, z)(s), p(z, y)(t)) \end{aligned}$$

whenever $s \leq \mu_1(x, z)$, $t \leq \mu_1(z, y)$ and $s + t \leq \mu_1(x, y)$.

(ii) Similarly, from (FPM4)(ii), we know that

$$R(p(x, y)(s + t - u), p(z, z)(u)) \leq R(p(x, z)(s), p(z, y)(t))$$

whenever $s \geq \mu_1(x, z)$, $t \geq \mu_1(z, y)$, $u \geq \mu_1(z, z)$ and $s + t - u \geq \mu_1(x, y)$. From here, we obtain

$$p(x, y)(s + t - u) \leq R(p(x, y)(s + t - u), p(z, z)(u)) \leq R(p(x, z)(s), p(z, y)(t))$$

whenever $s \geq \mu_1(x, z)$, $t \geq \mu_1(z, y)$, $u \geq \mu_1(z, z)$ and $s+t-u \geq \mu_1(x, y)$. Since $p(x, y) \in G$ is non-increasing on $[\mu_1(x, y), \infty)$, we have that

$$\begin{aligned} d_p(x, y)(s + t) &= p(x, y)(s + t) \leq p(x, y)(s + t - u) \leq R(p(x, y)(s + t - u), p(z, z)(u)) \\ &\geq R(p(x, z)(s), p(z, y)(t)) \end{aligned}$$

whenever $s \geq \mu_1(x, z)$, $t \geq \mu_1(z, y)$ and $s + t \geq \mu_1(x, y)$.

Now, assume that (X, p, L, R) is 0-complete and let (x_n) be a Cauchy sequence in (X, d_p, L, R). We may suppose that $x_n \neq x_m$ for all $n \neq m$ without lose of generality. Hence we have that $lim_{n,m\to\infty} d_p(x_n, x_m) = lim_{n,m\to\infty} p(x_n, x_m) = \overline{0}$ which means that (x_n) is 0-Cauchy sequence in (X, p, L, R). Since (X, p, L, R) is 0-complete there is a point $x \in X$ such that $lim_{n,m\to\infty} p(x_n, x_m) = lim_{n\to\infty} p(x_n, x) = p(x, x) = \overline{0}$. Thus we obtain $lim_{n\to\infty} d_p(x_n, x) = \overline{0}$ which follows that (X, d_p, L, R) is complete. The converse of this assertion can be shown with a similar procedure. □

Theorem 4.3. *Let (X, p, min, max) be a fuzzy partial metric space. Then the mapping $d^*_p : X \times X \to G$ defined by*

$$d^*_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a fuzzy metric on X. If (X, p, L, R) is complete, then (X, d^*_p, L, R) is complete.

Corollary 4.4. *If (X, p, min, max) is a fuzzy partial metric space such that $p(x, x) = \overline{a}$ $(a \in \mathbb{R})$ for all $x \in X$ and $p(x, x) \prec p(x, y)$ for all $x \neq y$, then the mapping $d^*_p : X \times X \to G$ defined by*

$$d^*_p(x, y) = p(x, y) - p(x, x)$$

is a fuzzy metric on X.

Theorem 4.5. *Let (X, q, L, R) be a fuzzy quasi metric space satisfying that L is a continuous t-norm and R is a continuous t-conorm. Then the mapping $d_q : X \times X \to G$ defined by*

$$d_q(x, y)(s + t) = max(q(x, y)(s), q(y, x)(t))$$

for all $x, y \in X$ and $s, t \geq 0$, is a fuzzy metric on X.

Proof. (FM1) Let $x = y$. Then, for all $s > 0$, we have

$$d_q(x, x)(s) = max(q(x, x)(\tfrac{s}{2}), q(x, x)(\tfrac{s}{2})) = q(x, x)(\tfrac{s}{2}) = 0.$$

If $s = 0$, then

$$d_q(x, x)(0) = max(q(x, x)(0), q(x, x)(0) = q(x, x)(0) = 1.$$

This follows that $d_q(x, x) = \bar{0}$. Now, suppose that $d_q(x, y) = \bar{0}$ for all $x, y \in X$. Then, for all $s > 0$, we have

$$d_q(x, y)(2s) = max(q(x, y)(s), q(y, x)(s)) = 0$$

which means that $q(x, y)(s) = 0$ and $q(y, x)(s)) = 0$. Since $q(x, y), q(y, x) \in G$, we obtain $q(x, y)(0) = 1$ and $q(y, x)(0) = 1$. This means that $q(x, y) = q(y, x) = \bar{0}$. So, it is seen that $x = y$.

(FM2) It is obvious from definition of d_q that $d_q(x, y) = d_q(y, x)$.

(FM3) Let $x, y, z \in X$.

(i) Suppose that $s \leq \mu_1(x, z)$, $t \leq \mu_1(z, y)$ and $s + t \leq \mu_1(x, y)$ where $[d_q(x, y)]_\alpha = [\mu_\alpha(x, y), \nu_\alpha(x, y)]$. Then, we obtain

$$\begin{aligned} d_q(x, y)(s + t) &= max(q(x, y)(s), q(y, x)(t)) \\ &\geq max(L(q(x, z)(s - t), q(z, y)(s)), L(q(y, z)(t - s), q(z, x)(t))) \\ &= L(max(q(x, z)(s - t), q(z, x)(t)), max(q(z, y)(s), q(y, z)(t - s))) \\ &= L(d_q(x, z)(s), d_q(z, y)(t)). \end{aligned}$$

(ii) Now, suppose that $s \geq \mu_1(x, z)$, $t \geq \mu_1(z, y)$ and $s + t \geq \mu_1(x, y)$. Then, we have

$$\begin{aligned} d_q(x, y)(s + t) &= max(q(x, y)(s), q(y, x)(t)) \\ &\leq max(R(q(x, z)(s - t), q(z, y)(s)), R(q(y, z)(t - s), q(z, x)(t))) \\ &= R(max(q(x, z)(s - t), q(z, x)(t)), max(q(z, y)(s), q(y, z)(t - s))) \\ &= R(d_q(x, z)(s), d_q(z, y)(t)). \end{aligned}$$

Hence, (X, d_q, L, R) is a fuzzy metric space whenever L and R are continuous mappings. □

Corollary 4.6. *If (X, p, min, max) is a fuzzy partial metric space, then the mapping $\widetilde{d_p} : X \times X \to G$ defined by*

$$\widetilde{d_p}(x, y)(s + t) = max(p(x, y)(s) - p(x, x)(s), p(x, y)(t) - p(y, y)(t))$$

for all $x, y \in X$ and $s, t \geq 0$, is a fuzzy metric on X.

As an application to the obtained results and properties, we give the Banach fixed point theorem in fuzzy partial metric spaces as follows:

Theorem 4.7. *Let (X, p, min, max) be a complete fuzzy partial metric space satisfying $lim_{t\to\infty} p(x, y)(t) = 0$ for all $x, y \in X$. If $T : X \to X$ is a mapping such that*

$$p(Tx, Ty) \preccurlyeq kp(x, y) \text{ for all } x, y \in X,$$

where $k \prec \bar{1}$ $(k \in G)$, then T has a unique fixed point in X.

Proof. If (X, p, min, max) is a complete fuzzy partial metric space satisfying $lim_{t\to\infty}p(x,y)(t) = 0$ for all $x, y \in X$, then by Theorem 4.2, (X, d_p, min, max) is a complete fuzzy metric space satisfying $lim_{t\to\infty}p(x,y)(t) = 0$ for all $x, y \in X$. Then, all assumptions of Theorem 4.3 in [17] are held and so we obtain that T has a unique fixed point in X. □

5. Topologies induced by a fuzzy partial metric

As we know from [20], if (X, P) is a crisp partial metric space, then we can induce a crisp topology (denoted by T_P) from this partial metric by taking the family $\{B_P(x,\varepsilon) : x \in X, \varepsilon > 0\}$ as a basis where $B_P(x,\varepsilon) = \{y|P(x,z) < P(x,x) + \varepsilon\}$ whenever $x \in X$ and $\varepsilon > 0$. In this section, with the above consideration, we first show that a crisp topology can be induced from a given fuzzy partial metric space and then we present that Lowen's fuzzy topology and fuzzifying topology can be obtained which are based on the level topologies.

Theorem 5.1. *Let (X, p, L, max) be a fuzzy partial metric space, $\varepsilon > 0$ and $\alpha \in (0, 1]$. Then the family $\{B^\nu_\alpha(x,\varepsilon)|x \in X, \varepsilon > 0\}$ of sets $B^\nu_\alpha(x,\varepsilon) = \{y|\nu_\alpha(x,y) < \nu_\alpha(x,x) + \varepsilon\}$ forms a basis for a topology on X and this topology is denoted by $(T_p)^\nu_\alpha$. i.e., $(T_p)^\nu_\alpha =< \{B^\nu_\alpha(x,\varepsilon)|x \in X, \varepsilon > 0\} >$.*

Proof. Let $\alpha \in (0, 1]$. Then $x \in B^\nu_\alpha(x,\varepsilon)$ for all $x \in X$ and $\varepsilon > 0$. This follows that $X = \bigcup_{\substack{x\in X,\\ \varepsilon>0}} B^\nu_\alpha(x,\varepsilon)$. Suppose that $B^\nu_\alpha(x,\varepsilon_1) \cap B^\nu_\alpha(y,\varepsilon_2) \neq \emptyset$ for any $x, y \in X$ and $\varepsilon_1, \varepsilon_2 > 0$. It means that there exists a point $a \in X$ such that $a \in B^\nu_\alpha(x,\varepsilon_1) \cap B^\nu_\alpha(y,\varepsilon_2)$. Choose $\varepsilon = min(\varepsilon_1 + \nu_\alpha(x,x) - \nu_\alpha(x,a), \varepsilon_2 + \nu_\alpha(y,y) - \nu_\alpha(y,a))$. Now, we claim that $B^\nu_\alpha(a,\varepsilon) \subseteq B^\nu_\alpha(x,\varepsilon_1) \cap B^\nu_\alpha(y,\varepsilon_2)$. Take $z \in B^\nu_\alpha(a,\varepsilon)$. Then, we have that

$$\begin{aligned}\nu_\alpha(x,z) &\leq \nu_\alpha(x,a) + \nu_\alpha(a,z) - \nu_\alpha(a,a) < \nu_\alpha(x,a) + \nu_\alpha(a,a) + \varepsilon - \nu_\alpha(a,a)\\ &< \nu_\alpha(x,a) + \varepsilon_1 + \nu_\alpha(x,x) - \nu_\alpha(x,a) = \nu_\alpha(x,x) + \varepsilon_1\end{aligned}$$

which means $z \in B^\nu_\alpha(x,\varepsilon_1)$. With the similar proceed, we can show that $z \in B^\nu_\alpha(y,\varepsilon_2)$. Hence, the family $\{B^\nu_\alpha(x,\varepsilon)|x \in X, \varepsilon > 0\}$ of sets $B^\nu_\alpha(x,\varepsilon) = \{y|\nu_\alpha(x,y) < \nu_\alpha(x,x)+\varepsilon\}$ is a basis for a topology on X. □

Theorem 5.2. *Let (X, p, min, R) be a fuzzy partial metric space, $\varepsilon > 0$ and $\alpha \in (0, 1]$. Then the family $\{B^\mu_\alpha(x,\varepsilon)|x \in X, \varepsilon > 0\}$ of sets $B^\mu_\alpha(x,\varepsilon) = \{y|\mu_\alpha(x,y) < \mu_\alpha(x,x) + \varepsilon\}$ forms a basis for a topology on X and this topology is denoted by $(T_p)^\mu_\alpha$. i.e., $(T_p)^\mu_\alpha =< \{B^\mu_\alpha(x,\varepsilon)|x \in X, \varepsilon > 0\} >$.*

Proof. The proof can be obtained similar to the proof of Theorem 5.1. □

Corollary 5.3. *If (X, p, min, max) is a fuzzy partial metric space, then $(T_p)^\mu_\alpha = (T_p)^\nu_\alpha$ for all $\alpha \in (0, 1]$.*

Proof. Let $z \in B^\nu_\alpha(x,\varepsilon)$ for any $x \in X$, $\varepsilon > 0$ and $\alpha \in (0, 1]$. Take $\varepsilon^* = \nu_\alpha(x,x) - \mu_\alpha(x,x) + \varepsilon > 0$. Then

$$\mu_\alpha(x,z) \leq \nu_\alpha(x,z) < \nu_\alpha(x,x) + \varepsilon = \nu_\alpha(x,x) + \varepsilon^* - \nu_\alpha(x,x) + \mu_\alpha(x,x) = \mu_\alpha(x,x) + \varepsilon^*.$$

Hence, we obtain $z \in B^\nu_\alpha(x,\varepsilon^*)$ which means that $B^\nu_\alpha(x,\varepsilon) \subset B^\nu_\alpha(x,\varepsilon^*)$. Now, assume that $z \in B^\mu_\alpha(x,\varepsilon)$ for any $x \in X$, $\varepsilon > 0$ and $\alpha \in (0, 1]$. By choosing $\varepsilon^* = \nu^(_\alpha x,z) - \mu_\alpha(x,z) + \varepsilon > 0$, we obtain

$$\mu_\alpha(x,z) < \mu_\alpha(x,x) + \varepsilon \leq \nu_\alpha(x,x) + \varepsilon = \nu_\alpha(x,x) + \varepsilon^* - \nu_\alpha(x,z) + \mu_\alpha(x,z)$$

which follows that $\nu_\alpha(x,z) < \nu_\alpha(x,x) + \varepsilon^*$. This means that $z \in B^\nu_\alpha(x,\varepsilon^*)$ and so, we obtain that $B^\mu_\alpha(x,\varepsilon) \subset B^\nu_\alpha(x,\varepsilon^*)$. □

Proposition 5.4. *If (X,P) is a crisp metric space and (X,p,min,max) is a fuzzy partial metric space constructed as given in Corollary 3.7, then we obtain that $T_P = (T_p)^\mu_\alpha = (T_p)^\nu_\alpha$.*

Proof. This proof can be completed with the similar process as given above by taking attention $[p(x,y)]_\alpha = \{t|p(x,y)(t) \geq \alpha\} = \{P(x,y)\}$ for all $x,y \in X$ and $\alpha \in (0,1]$. □

Remark 5.5. The topological space induced by a fuzzy partial metric may not admit a compatible fuzzy metric as seen by taking the example given in [14] since a crisp topological space (X,T) is metrizable if and only if it admits a compatible fuzzy metric.

In the next example, we show that the topologies induced by a fuzzy partial metric are not coincident with that induced by the fuzzy metric induced by a fuzzy partial metric.

Example 5.6. Consider the partial metric space $(\mathbb{R}^+, p_{max}, min, max)$ given in Example 3.2 and the fuzzy metric $(\mathbb{R}^+, d_{p_{max}}, min, max)$ given in Theorem 4.2 where

$$d_{p_{max}}(x,y) = \begin{cases} p_{max}(x,y), & x \neq y \\ \overline{0}, & x = y \end{cases}.$$

Here, we obtain that $[p_{max}(x,y)]_\alpha = \{t : \overline{max\{x,y\}}(t) \geq \alpha\} = \{max\{x,y\}\}$ for all $\alpha \in (0,1]$. Also, with a simple calculation, we have $B^\mu_\alpha(1,\frac{1}{4}) = [0,\frac{5}{4})$ and $N_1(\frac{1}{4},\alpha) = \{1\}$. Since we can not find any α to satisfy $[0,\frac{5}{4}) \subseteq \{1\}$, we conclude that the topologies $(T_{p_{max}})^\mu_\alpha = (T_{p_{max}})^\nu_\alpha$ induced from fuzzy partial metric p_{max} and $T_{d_{p_{max}}}$ induced from fuzzy metric $d_{p_{max}}$ are not same.

As a continuation, we obtain the relations between the topologies induced by a fuzzy partial metric and that induced by the fuzzy metric induced by a fuzzy partial metric as follows:

Proposition 5.7. *If (X,p,min,max) is a fuzzy partial metric space, then we have the followings:*
(i) $(T_p)^\mu_\alpha = (T_p)^\nu_\alpha \subseteq T_{d_p}$,
(ii) $(T_p)^\mu_\alpha = (T_p)^\nu_\alpha = T_{d^*_p}$ if $p(x,x) = \overline{a}$ $(a \in \mathbb{R})$ for all $x \in X$,
(iii) $T_{d^*_p} \subset T_{d_p}$.

Proof. (ii) If $[p(x,y)]_\alpha = [\mu_\alpha(x,y), \nu_\alpha(x,y)]$ and $[p(x,x)]_\alpha = [\mu_\alpha(x,x), \nu_\alpha(x,x)]$, then by Lemma 2.3(ii), we have $[d^*_p(x,y)]_\alpha = [\mu_\alpha(x,y) - \nu_\alpha(x,x), \nu_\alpha(x,y) - \mu_\alpha(x,x)]$. Take $a \in B^\nu_\alpha(x,\varepsilon)$ for $x \in X$ and $\varepsilon > 0$. Then, $\nu_\alpha(x,a) < \nu_\alpha(x,x) + \varepsilon$. Since $p(x,x) = \overline{a}$ $(a \in \mathbb{R})$ for all $x \in X$, $\mu_\alpha(x,x) = \nu_\alpha(x,x) = a$. This follows that $\nu_\alpha(x,a) < \mu_\alpha(x,x) + \varepsilon$ which means $a \in (B_{d^*_p})^\nu_\alpha(x,\varepsilon)$. Thus, $T_{d^*_p} \subseteq (T_p)^\mu_\alpha = (T_p)^\nu_\alpha$. With the similar way, the converse can be shown. □

Corollary 5.8. *If (X,p,min,max) is a fuzzy partial metric space such that $p(x,x) = \overline{a}$ $(a \in \mathbb{R})$ for all $x \in X$ and $p(x,x) \prec p(x,y)$ for all $x \neq y$, then $(X,(T_p)^\mu_\alpha)$ (or $(X,(T_p)^\nu_\alpha)$) is a Hausdorff space. Moreover, $(X,(T_p)^\mu_\alpha)$ (or $(X,(T_p)^\nu_\alpha)$) is metrizable.*

In the following theorems, we show that α-level topologies induced by a fuzzy partial metric space can construct a Lowen fuzzy topology.

Theorem 5.9. *Let (X,p,L,max) be a fuzzy partial metric space and $\{(T_p)^\nu_\alpha|\alpha \in (0,1]\}$ be the family of topologies induced by this fuzzy partial metric. Then the family of fuzzy sets*

$$\tau^\nu_p = \{x|[x]_\alpha \in (T_p)^\nu_\alpha,\ \forall \alpha \in (0,1]\}$$

is a Lowen fuzzy topology on X.

Proof. (L1) Let $\alpha \in (0,1]$. If $a \geq \alpha$, then $[\underline{a}]_\alpha = X \in (T_p)^\nu_\alpha$. Otherwise, if $a < \alpha$, then $[\underline{a}]_\alpha = \emptyset \in (T_p)^\nu_\alpha$. Thus, we have $\underline{a} \in \tau^\nu_p$.

(L2) Let $f_1, f_2 \in \tau^\nu_p$. Then, $[f_1]_\alpha \in (T_p)^\nu_\alpha$ and $[f_2]_\alpha \in (T_p)^\nu_\alpha$ for all $\alpha \in (0,1]$. Since $(T_p)^\nu_\alpha$ is a topology on X for all $\alpha \in (0,1]$, $[f_1]_\alpha \cap [f_2]_\alpha \in (T_p)^\nu_\alpha$ is obtained. Hence, we get $f_1 \vee f_2 \in \tau^\nu_p$.

(L3) Let $f_i \in \tau^\nu_p$ for all $i \in J$. Then $[f_i]_\alpha \in (T_p)^\nu_\alpha$ for all $\alpha \in (0,1]$. Since $(T_p)^\nu_\alpha$ is a topology on X for all $\alpha \in (0,1]$, it follows that $\bigcup_{i\in J}[f_i]_\alpha \in (T_p)^\nu_\alpha$. Hence, we obtain that $\bigvee_{i\in J} f_i \in \tau^\nu_p$. □

Theorem 5.10. *Let (X, p, min, R) be a fuzzy partial metric space and $\{(T_p)^\mu_\alpha | \alpha \in (0,1]\}$ be the family of topologies induced by this fuzzy partial metric. Then the family of fuzzy sets*

$$\tau^\mu_p = \{x | [x]_\alpha \in (T_p)^\mu_\alpha,\ \forall \alpha \in (0,1]\}$$

is a Lowen fuzzy topology on X.

Proof. The proof can be obtained similar to the proof of Theorem 5.9. □

Corollary 5.11. *If (X, p, min, max) is a fuzzy partial metric space, then $\tau^\mu_p = \tau^\nu_p$.*

Now, we give the definitions of $\beta - \nu$-open ball and $\beta - \mu$-open ball to construct the basis for a Lowen fuzzy topology.

Definition 5.12. (1) Let (X, p, L, max) be a fuzzy partial metric space, $x \in X$, $\varepsilon > 0$ and $\alpha, \beta \in (0,1]$. Then the fuzzy set $\beta B^\nu_\alpha(x, \varepsilon)$ defined by

$$\beta B^\nu_\alpha(x,\varepsilon)(y) = \begin{cases} \beta, & y \in B^\nu_\alpha(x,\varepsilon) \\ 0, & \text{otherwise} \end{cases}$$

is called $\beta - \nu$-open ball centered at x with radius ε.

(2) Let (X, p, min, R) be a fuzzy partial metric space, $x \in X$, $\varepsilon > 0$ and $\alpha, \beta \in (0,1]$. Then the fuzzy set $\beta B^\mu_\alpha(x, \varepsilon)$ defined by

$$\beta B^\mu_\alpha(x,\varepsilon)(y) = \begin{cases} \beta, & y \in B^\mu_\alpha(x,\varepsilon) \\ 0, & \text{otherwise} \end{cases}$$

is called $\beta - \mu$-open ball centered at x with radius ε.

Remark 5.13. If (X, p, min, max) is a fuzzy partial metric space, then the $\beta - \mu$-open balls are coincident with the $\beta - \nu$-open balls.

Theorem 5.14. *Let (X, p, L, max) be a fuzzy partial metric space. Then the family of fuzzy sets*

$$\mathfrak{B}_1 = \{\beta B^\nu_\alpha(x,\varepsilon) | x \in X, \varepsilon > 0, \alpha \in (0,1], \beta \in [\alpha, 1]\}$$

is a basis for the Lowen fuzzy topology τ^ν_p.

Proof. We first show that $\mathfrak{B}_1 \subseteq \tau^\nu_p$. Take $\beta B^\nu_\alpha(x,\varepsilon) \in \mathfrak{B}_1$ for any $x \in X, \varepsilon > 0, \alpha \in (0,1], \beta \in [\alpha, 1]$. Since $\alpha \leq \beta$, then we have $[\beta B^\nu_\alpha(x,\varepsilon)]_\alpha = B^\nu_\alpha(x,\varepsilon) \in (T^\nu_p)_\alpha$. This follows that $\beta B^\nu_\alpha(x,\varepsilon) \in \tau^\nu_p$. Now, assume that $f \in \tau^\nu_p$ and $f(x) > 0$. Then $[f]_\alpha \in (T^\nu_p)_\alpha$ for all $f(x) \geq \alpha$ whenever $\alpha \in (0,1]$. Thus $[f]_\alpha \in (T^\nu_p)_\alpha$ whenever $x \in [f]_\alpha$. From the definition of $(T^\nu_p)_\alpha$, there exists $\varepsilon_1 > 0$ such that $B^\nu_\alpha(x,\varepsilon_1) \subset [f]_\alpha$. It means that $f(y) \geq \alpha$ for all $y \in B^\nu_\alpha(x,\varepsilon_1)$. Hence, we obtain that $\alpha B^\nu_\alpha(x,\varepsilon_1) \leq f$. □

Theorem 5.15. *Let (X, p, min, R) be a fuzzy partial metric space. Then the family of fuzzy sets*

$$\mathfrak{B}_2 = \{\beta B^\mu_\alpha(x,\varepsilon) | x \in X, \varepsilon > 0, \alpha \in (0,1], \beta \in [\alpha, 1]\}$$

is a basis for the Lowen fuzzy topology τ^μ_p.

Proof. The proof can be obtained similar to the proof of Theorem 5.14.

Finally, we show that a fuzzifying topology can be induced from a given fuzzy partial metric space with the processes of which one is based on the level topology and the other one is direct. □

Theorem 5.16. *Let (X, p, L, max) be a fuzzy partial metric space such that $p(x,x) = \overline{a}$ $(a \in \mathbb{R})$ and $\{(T_p)^{\nu}_{\alpha} | \alpha \in (0,1]\}$ be the family of topologies induced by this fuzzy partial metric. Then the mapping $\tau_p : 2^X \to [0,1]$ defined by*

$$\tau_p(A) = sup\{\alpha \in (0,1] | A \in (T^{\nu}_p)_{\alpha}\}$$

is a fuzzifying topology on X.

Proof. Let $\alpha_1 < \alpha_2$. To show $(T^{\nu}_p)_{\alpha_2} \subseteq (T^{\nu}_p)_{\alpha_1}$, let $y \in B^{\nu}_{\alpha_1}(x,\varepsilon)$ whenever $x \in X$ and $\varepsilon > 0$. Then, $\nu_{\alpha_1}(x,y) < \nu_{\alpha_1}(x,x) + \varepsilon$. Since, ν_{α} is non-increasing with respect to α, the following inequality is obtained:

$$\nu_{\alpha_2}(x,y) \leq \nu_{\alpha_1}(x,y) < \nu_{\alpha_1}(x,x) + \varepsilon = \nu_{\alpha_2}(x,x) + \varepsilon.$$

This means that $y \in B^{\nu}_{\alpha_2}(x,\varepsilon)$ and so we have that $(T^{\nu}_p)_{\alpha_2} \subseteq (T^{\nu}_p)_{\alpha_1}$. Thus, the family $\{(T_p)^{\nu}_{\alpha} | \alpha \in (0,1]\}$ is non-decreasing with respect to α. Hence, we have, by Lemma 2.3 (in [22]), τ_p is a fuzzifying topology on X. □

Theorem 5.17. *Let (X, p, min, max) be a fuzzy partial metric space and define the mapping $N^p_x : 2^X \to [0,1]$ by*

$$N^p_x(A) = \bigvee_{t>0} \bigwedge_{y \notin A} p(x,y)(t)$$

satisfies the following properties (whose implies the element of partial generalized neighborhood system $\mathcal{N} = \{N^p_x : x \in X\}$ given in [32]):

(PGN1) $N^p_x(X) = 1$,
(PGN2) If $A \subset B$, then $N^p_x(A) \leq N^p_x(B)$,
(PGN3) $N^p_x(A) \cap N^p_x(B) \leq N^p_x(A \wedge B)$,
(PGN4) If $x \neq A$, then $N^p_x(A) = N^p_x(\emptyset)$,
(PGN5) $N^p_x(A) = \bigvee_{B \subseteq A}(N^p_x(B) \wedge \bigwedge_{y \in B} N^p_y(A))$.

Also, according to [32], the mapping $\tau' : 2^X \to [0,1]$ defined by $\tau'_p(A) = \bigwedge_{x \in A} N^p_x(A)$ is a fuzzifying topology on X.

Proof. The proof can be completed with the similar process given in [32]. □

Remark 5.18. We note that the fuzzifying topologies τ_p and τ'_p are coincident according to the study given in [34].

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